Dear Author,

Here are the proofs of your article.

- You can submit your corrections **online**, via **e-mail** or by **fax**.
- For **online** submission please insert your corrections in the online correction form. Always indicate the line number to which the correction refers.
- You can also insert your corrections in the proof PDF and **email** the annotated PDF.
- For fax submission, please ensure that your corrections are clearly legible. Use a fine black pen and write the correction in the margin, not too close to the edge of the page.
- Remember to note the **journal title**, **article number**, and **your name** when sending your response via e-mail or fax.
- **Check** the metadata sheet to make sure that the header information, especially author names and the corresponding affiliations are correctly shown.
- **Check** the questions that may have arisen during copy editing and insert your answers/ corrections.
- **Check** that the text is complete and that all figures, tables and their legends are included. Also check the accuracy of special characters, equations, and electronic supplementary material if applicable. If necessary refer to the *Edited manuscript*.
- The publication of inaccurate data such as dosages and units can have serious consequences. Please take particular care that all such details are correct.
- Please **do not** make changes that involve only matters of style. We have generally introduced forms that follow the journal's style.
  Substantial changes in content, e.g., new results, corrected values, title and authorship are not allowed without the approval of the responsible editor. In such a case, please contact the Editorial Office and return his/her consent together with the proof.
- If we do not receive your corrections **within 48 hours**, we will send you a reminder.
- Your article will be published **Online First** approximately one week after receipt of your corrected proofs. This is the **official first publication** citable with the DOI. **Further changes are, therefore, not possible.**
- The **printed version** will follow in a forthcoming issue.

Please note

After online publication, subscribers (personal/institutional) to this journal will have access to the complete article via the DOI using the URL: http://dx.doi.org/[DOI].
If you would like to know when your article has been published online, take advantage of our free alert service. For registration and further information go to: http://www.link.springer.com.

Due to the electronic nature of the procedure, the manuscript and the original figures will only be returned to you on special request. When you return your corrections, please inform us if you would like to have these documents returned.

Metadata of the article that will be visualized in OnlineFirst

Please note: Images will appear in color online but will be printed in black and white.

ArticleTitle	A descent algorithm for generalized complementarity problems based on generalized Fischer-Burmeister functions	
Article Sub-Title		
Article CopyRight	SBMAC - Sociedade Brasileira de Matemática Aplicada e Computacional (This will be the copyright line in the final PDF)	
Journal Name	Computational and Applied Mathematics	

Corresponding Author	Family Name	**Tawhid**
	Particle	
	Given Name	**Mohamed A.**
	Suffix	
	Division	Department of Mathematics and Statistics, Faculty of Science
	Organization	Thompson Rivers University
	Address	V2C 0C8, Kamloops, BC, Canada
	Division	Department of Mathematics and Computer Science, Faculty of Science
	Organization	Alexandria University
	Address	Moharam Bey, 21511, Alexandria, Egypt
	Email	mtawhid@tru.ca
Author	Family Name	**Gu**
	Particle	
	Given Name	**Wei-Zhe**
	Suffix	
	Division	Department of Mathematics and Statistics, Faculty of Science
	Organization	Thompson Rivers University
	Address	V2C 0C8, Kamloops, BC, Canada
	Division	Department of Mathematics, School of Science
	Organization	Tianjin University
	Address	300072, Tianjin, People’s Republic of China
	Email	guweizhe@tju.edu.cn
Author	Family Name	**Tran**
	Particle	
	Given Name	**Benjamin**
	Suffix	
	Division	Department of Mathematics and Statistics, Faculty of Science
	Organization	Thompson Rivers University
	Address	V2C 0C8, Kamloops, BC, Canada
	Email	

	Received	29 May 2014
Schedule	Revised	13 February 2016
	Accepted	17 February 2016

Abstract	We study an unconstrained minimization approach to the generalized complementarity problem GCP(*f*, *g*) based on the generalized Fischer-Burmeister function and its generalizations when the underlying functions are C^1. Also, we show how, under appropriate regularity conditions, minimizing the merit function corresponding to *f* and *g* leads to a solution of the generalized complementarity problem. Moreover, we propose a descent algorithm for GCP(*f*, *g*) and show a result on the global convergence of a descent algorithm for solving generalized complementarity problem. Finally, we present some preliminary numerical results. Our results further give a unified/generalization treatment of such results for the nonlinear complementarity problem based on generalized Fischer-Burmeister function and its generalizations.
Keywords (separated by '-')	Generalized complementarity problem - GCP function - Generalized FB function - Unconstrained minimization - Regularity conditions - Merit function - Descent algorithm
Mathematics Subject Classification (separated by '-')	90C33 - 90C56
Footnote Information	Communicated by Ernesto G. Birgin.

Comp. Appl. Math.
DOI 10.1007/s40314-016-0328-6

Author Proof

A descent algorithm for generalized complementarity problems based on generalized Fischer-Burmeister functions

Mohamed A. Tawhid[1,2] · **Wei-Zhe Gu**[1,3] · **Benjamin Tran**[1]

Received: 29 May 2014 / Revised: 13 February 2016 / Accepted: 17 February 2016
© SBMAC - Sociedade Brasileira de Matemática Aplicada e Computacional 2016

Abstract We study an unconstrained minimization approach to the generalized complementarity problem GCP(f, g) based on the generalized Fischer-Burmeister function and its generalizations when the underlying functions are C^1. Also, we show how, under appropriate regularity conditions, minimizing the merit function corresponding to f and g leads to a solution of the generalized complementarity problem. Moreover, we propose a descent algorithm for GCP(f, g) and show a result on the global convergence of a descent algorithm for solving generalized complementarity problem. Finally, we present some preliminary numerical results. Our results further give a unified/generalization treatment of such results for the nonlinear complementarity problem based on generalized Fischer-Burmeister function and its generalizations.

Keywords Generalized complementarity problem · GCP function · Generalized FB function · Unconstrained minimization · Regularity conditions · Merit function · Descent algorithm

Mathematics Subject Classification 90C33 · 90C56

uncorrected proof

Communicated by Ernesto G. Birgin.

✉ Mohamed A. Tawhid
mtawhid@tru.ca

Wei-Zhe Gu
guweizhe@tju.edu.cn

1 Department of Mathematics and Statistics, Faculty of Science, Thompson Rivers University, Kamloops V2C 0C8, BC, Canada

2 Department of Mathematics and Computer Science, Faculty of Science, Alexandria University, Moharam Bey, Alexandria 21511, Egypt

3 Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, People's Republic of China

Author Proof

1 Introduction

Consider the generalized complementarity problem corresponding to f and g, denoted by GCP(f, g), which is to find a vector $x^* \in \Re^n$ such that

$$f(x^*) \geq 0, \quad g(x^*) \geq 0 \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle = 0 \tag{1.1}$$

where $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are given C^1 functions.

Many researchers have studied the above formulation of GCP(f, g), its numerical methods, and applications. See Hyer et al. (1997), Isac (1992), Noor (1993) and the references cited therein. Also GCP(f, g) covers some related problems studied in the literature in the last decades; for example, GCP(f, g) reduces to the nonlinear complementarity problem NCP(f) when $g(x) = x$. By taking in NCP(f) $f(x) = Mx + q$ with $M \in R^{n\times n}$ and a vector $q \in R^n$, then NCP(f) is called a linear complementarity problem LCP(M, q). Also, GCP(f, g) is known as the quasi/implicit complementarity problem when $g(x) = x - W(x)$ with some $W : R^n \to R^n$, see, e.g., Isac (1992), Noor (1988), Pang (1981).

The importance of these problems in operations research, optimization, engineering sciences, economics and other areas has been well documented in the literature, see e.g., Cottle et al. (1992), Cottle et al. (1980), Ferris and Pang (1997a, b), Harker and Pang (1990), Di Pillo and Giannessi (1996), and the references therein.

1.1 Example applications

1.1.1 Traffic equilibrium problem with nonadditive costs

The study of traffic equilibrium problem (TEP) has witnessed a growing amount of research attentions recently as researchers have presented various formulations in which many different assumptions are made to represent the real traffic conditions (see., e.g., Aashtiani and Magnanti 1981; Chen et al. 1999). One of the standard assumptions in these studies is the additivity of route cost. That is, the route cost is simply the sum of the link costs on that route. There are many studies about TEP with additive route costs assumptions, a detailed overview can refer to Patriksson (1994, 2004), Sheffi (1985).

There are many situations, however, where this additivity assumption on the route costs is inappropriate. In Gabriel and Bernstein (1997), the authors discussed some of the situations where nonadditive route costs occur. They claimed that almost all toll and fare schemes being implemented around the world are nonadditive. For example, the different pricing policies such as congestion pricing and the collection of emission fees add to the nonadditivity of travel costs. Moreover, different individuals have different valuations of time, which contributes to the nonadditivity of route costs. Although nonadditivity is important in presenting a more realistic view of the traffic situation, it causes a difficulty in the analysis and computation of an equilibrium, which are usually done by formulating the TEP as the variational inequality problem (VIP). Special cases of the VIP include the nonlinear complementarity problem (NCP). The TEP with additive costs may be formulated as a monotone VIP (see, Facchinei and Pang 2003). In Lo and Chen (2000), the authors studied a special case of the TEP with nonadditive cost functions. In particular, under the assumption the route cost is the sum of the travel time and an additional charge which is route specific (a specific travel cost, possibly in the form of toll, is added only to a particular route in the network), they introduced a route-specific cost structure where this additional cost is only incurred by travelers on that route. They formulated TEP as a monotone NCP. Under other assumptions, TEP formulated as generalized complementarity problems see, e.g., Agdeppa et al. (2007), Xu and Gao (2011).

uncorrected proof

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

1.1.2 American options pricing

The real options approach has become a workhorse in modern economics and finance. However, many real options studies have focused on relatively simple option models. While these types of models have been successful in literature, real problems may involve more complex and realistic situations.

American options are contracts allowing the holder the right to sell (buy) an asset at a certain price at any time until a prespecified future date. The pricing of American options plays an important role both in theory and in real derivative markets. The American option pricing problem can be posed either as a linear complementarity problem (LCP) or a free boundary value problem (Company et al. 2014; McKean 1965; van Moerbeke 1976). These two different formulations have led to different methods for solving American options. The most algebraic approach of LCPs for American option pricing can be found in Feng et al. (2011), Huang and Pang (1998), Wilmott et al. (1995) and the references therein.

Most options traded on option exchanges worldwide and a large fraction of options traded over-the-counter are of the American style. These include options on stocks of individual companies, stock indexes, foreign currencies, interest rates, commodities, and energy. Options books of a large financial institution may contain options on thousands of different underlying assets, and perhaps several dozen different contracts (with expiration dates ranging from days to years, and different strike prices). As the underlying asset prices change throughout the trading day, the options prices change as well. Re-pricing a large options book in real time may thus require re-computing thousands of option prices quickly. For such large scale applications, fast numerical algorithms are essential. When the prices of underlying assets are assumed to follow a diffusion process, such as in the classical Black-Scholes-Merton model based on the geometric Brownian motion process, or in extensions such as Heston's stochastic volatility model, the pricing function of an American-style option solves a parabolic variational inequality. After this system is discretized in space and time, it yields a linear complementarity problem, which must be solved at each time step. Thus, the fast solution of linear complementarity problems (LCPs) is of great practical importance in computational finance. The standard treatment of LCPs for American options pricing can be found, for example, in Wilmott et al. (1995) for the simple case of the Black-Scholes-Merton model and in Huang and Pang (1998), Feng et al. (2011), Wilmott et al. (1995) and the references therein for several more complicated settings.

1.2 Motivation and outline

In the context of nonlinear complementarity problem (NCP), one of the well-known approaches to solve the NCP is to reformulate the original NCP as an unconstrained minimization problem whose global minima coincide with the solution of the NCP and the objective function of this unconstrained minimization problem is called a merit function for the NCP, (Facchinei and Kanzow 1997; Facchinei and Soares 1997; Fischer 1998, 1997; Geiger and Kanzow 1996; Jiang 1996; Kanzow 1996; Luca et al. 1996; Mangasarian and Solodov 1993; Yamashita and Fukushima 1995; Jein-shan Chen 2007; Chen 2006; Chen and Pan 2008; Chen et al. 2011). Most of the merit functions in these references are based on the square Fischer-Burmeister function (Facchinei and Soares 1997; Fischer 1998; Geiger and Kanzow 1996; Jiang 1996; Kanzow 1996; Luca et al. 1996), the implicit Lagrangian function (Jiang 1996; Mangasarian and Solodov 1993; Yamashita and Fukushima 1995), and generalized Fischer-Burmeister (Jein-shan Chen 2007; Chen 2006; Chen and Pan 2008; Chen et al. 2011). For other merit functions on the basis of various NCP functions, we refer

uncorrected proof

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

the interested reader to Galántai (2012) and the references therein. Most of these methods rely on the a so-called NCP function. In this paper, we follow this approach for generalized complementarity problem GCP(f, g) based on the generalized Fischer-Burmeister function. But first, we need to define GCP functions. A function $\phi : R^2 \to R$ is called a NCP function if it satisfies

$$\phi(a, b) = 0 \Leftrightarrow ab = 0, \quad a \geq 0,\ b \geq 0.$$

We call

$$\Phi(x) = \begin{bmatrix} \phi(f_1(x), g_1(x)) \\ \vdots \\ \phi(f_i(x), g_i(x)) \\ \vdots \\ \phi(f_n(x), g_n(x)) \end{bmatrix} \tag{1.2}$$

a GCP function for GCP(f, g). Solving for $\Phi(x) = 0$ is equivalent to finding the solution to the original problem. Then the function $\Psi : R^n \to R_+$ defined by

$$\Psi(x) := \frac{1}{2}\|\Phi(x)\|^2. \tag{1.3}$$

is a merit function for the GCP, i.e., the GCP can be recast as an unconstrained minimization:

$$\min_{x \in R^n} \Psi(x). \tag{1.4}$$

1.3 Example of GCP functions

Over the past two decades, a variety of NCP functions have been studied, see Galántai (2012) and references therein. Among which, some families of NCP functions (Chen and Pan 2008; Jein-shan Chen 2007; Hu et al. 2009) based on the Fischer-Burmeister function with p-norm are proposed. We give some examples of GCP functions based on these NCP functions.

1

Example 1 Suppose that f and g are C^1. Consider the following GCP function which is the basis of

$$\phi_p(a, b) := \|(a, b)\|_p - (a + b) \tag{1.5}$$

where p is any fixed real number in the interval $(1, +\infty)$ and $\|(a, b)\|_p$ denotes the p-norm of (a, b), i.e., $\|(a, b)\|_p = \sqrt[p]{|a|^p + |b|^p}$. The function ϕ_p was noted by Tseng (1996). For further study on this family of NCP functions, see Chen and Pan (2008).

The ith component of GCP function $\Phi(x)$ in (1.2) is defined as

$$\Phi_i(x) = \phi_p(f_i(x), g_i(x)) := f_i(x) + g_i(x) - \|(f_i(x), g_i(x))\|_p$$

Example 2 Consider the following GCP function which is based on proposed family of NCP functions (Chen and Pan 2008) relying on ϕ_p in (1.5) and some introduced NCP functions in Jein-shan Chen (2007):

$$\phi_1(a, b) := \phi_p(a, b) + \alpha a_+ b_+, \quad \alpha > 0 \tag{1.6}$$

where a_+ is defined as $\max(a, 0)$ and the ith component of GCP function $\Phi(x)$ in (1.2) is defined as

$$\Phi_i(x) = \phi_1(f_i(x), g_i(x)) := \phi_p(f_i(x), g_i(x)) + \alpha f_i(x)_+ g_i(x)_+, \quad \alpha > 0.$$

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Example 3 The following GCP function is based on NCP function in Chen and Pan (2008)

$$\phi_2(a,b) := \phi_p(a,b) + \alpha(ab)_+, \quad \alpha > 0. \tag{1.7}$$

We define the ith component of GCP function $\Phi(x)$ in (1.2) as

$$\Phi_i(x) = \phi_2(f_i(x), g_i(x)) := \phi_p(f_i(x), g_i(x)) + \alpha(f_i(x)\, g_i(x))_+, \quad \alpha > 0.$$

Example 4 The following GCP function is based on NCP function in Chen and Pan (2008)

$$\phi_3(a,b) := \sqrt{[\phi_p(a,b)]^2 + \alpha(a_+ b_+)^2}, \quad \alpha > 0. \tag{1.8}$$

We define the ith component of GCP function $\Phi(x)$ in (1.2) as

$$\Phi_i(x) = \phi_3(f_i(x), g_i(x)) := \sqrt{[\phi_p(f_i(x), g_i(x))]^2 + \alpha(f_i(x)_+\, g_i(x)_+)^2}, \quad \alpha > 0.$$

Example 5 We consider the following GCP function based on the NCP function in Chen and Pan (2008)

$$\phi_4(a,b) := \sqrt{[\phi_p(a,b)]^2 + \alpha[(ab)_+]^2}, \quad \alpha > 0. \tag{1.9}$$

The ith component of GCP function $\Phi(x)$ in (1.2) is defined as

$$\Phi_i(x) = \phi_4(f_i(x), g_i(x)) := \sqrt{[\phi_p(f_i(x), g_i(x))]^2 + \alpha[(f_i(x)\, g_i(x))_+]^2}, \quad \alpha > 0.$$

Example 6 We consider the following GCP function which is based on another family of NCP functions (Hu et al. 2009)

$$\phi_{\theta,p}(a,b) := a + b - \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a-b|^p}, \quad \theta \in (0,1]. \tag{1.10}$$

When $\theta = 1$, (1.10) will reduce to (1.5), i.e.,

$$\phi_{1,p}(a,b) = \phi_p(a,b) = a + b - \|(a,b)\|_p.$$

The ith component of GCP function $\Phi(x)$ in (1.2) is defined as

$$\begin{aligned}\Phi_i(x) &= \phi_{\theta,p}(f_i(x), g_i(x)) \\ &:= f_i(x) + g_i(x) - \sqrt[p]{\theta(|f_i(x)|^p + |g_i(x)|^p) + (1-\theta)|f_i(x) - g_i(x)|^p}, \\ &\quad\ \theta \in (0,1].\end{aligned}$$

Example 7 Based on (1.10) and NCP function in Chen and Pan (2008),

$$\phi_{\alpha,\theta,p}(a,b) := \frac{\alpha}{2}[(ab)_+]^2 + \frac{1}{2}\,[\phi_{\theta,p}(a,b)]^2, \quad \alpha \geq 0 \tag{1.11}$$

where $\phi_{\alpha,\theta,p}(a,b) : R^2 \to R_+$, we consider the ith component of GCP function $\Phi(x)$ in (1.2) as

$$\phi_{\alpha,\theta,p}(f_i(x), g_i(x)) := \frac{\alpha}{2}[(f_i(x)\, g_i(x))_+]^2 + \frac{1}{2}\,[\phi_{\theta,p}(f_i(x), g_i(x))]^2, \ \ \alpha \geq 0.$$

In this article, starting with C^1 functions f and g, we consider a generalized complementarity problem GCP(f,g) based on the generalized Fischer-Burmeister function. We consider a GCP function $\Phi : R^n \to R^n$ associated with GCP(f,g) and its merit function Ψ so that

$$\bar{x} \text{ solves GCP}(f,g) \ \Leftrightarrow \Phi(\bar{x}) = 0 \Leftrightarrow \Psi(\bar{x}) = 0.$$

uncorrected proof

In this paper, we show how, under appropriate regularity and strictly regularity conditions, finding local/global minimum of Ψ (or a 'stationary point' of Ψ) leads to a solution of the given generalized complementarity problem. Further, we show that how our results unify/extend various similar results proved in the literature for nonlinear complementarity problem when the underlying function is C^1. Moreover, we suggest a descent algorithm for GCP(f, g) and prove a result on the global convergence of a descent algorithm for solving generalized complementarity problem. Finally, we give some preliminary numerical results.

2 Preliminaries

Few words about notation. Throughout the paper, vector inequalities are interpreted componentwise. Vectors in R^n are regarded as column vectors. The inner-product between two vectors x and y in R^n is denoted by either $x^T y$ or $\langle x, y\rangle$. For a matrix A, the ith row of A is denoted by A_i. For a differentiable function $f : R^n \to R^m$, the Jacobian matrix of f at $\bar{x}$ is denoted by $\nabla f(\bar{x})$. The p-norm of x is denoted $||x||_p$ and the Euclidean norm of x is denoted by $||x||$. $\nabla_a \phi(a, b)$ and $\nabla_b \phi(a, b)$ denote the partial derivatives of ϕ with respect to the first variable and the second variable, respectively.

The author in Tawhid (2008) introduced the concepts of relatively monotonicity, $\mathbf{P_0}$-property and their variants for functions to minimize nonsmooth generalized complementarity problem under certain conditions.

Now we recall the following definitions from Tawhid (2008).

Definition 2.1 For functions $f, g : \Re^n \to \Re^n$, we say that f and g are:

(a) Relatively monotone if

$$\langle f(x) - f(y), g(x) - g(y)\rangle \geq 0 \quad \text{for all } x, y \in \Re^n.$$

(b) Relatively strictly monotone if

$$\langle f(x) - f(y), g(x) - g(y)\rangle > 0 \quad \text{for all } x, y \in \Re^n.$$

(c) Relatively strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle f(x) - f(y), g(x) - g(y)\rangle \geq \mu \|x - y\|^2 \quad \text{for all } x, y \in \Re^n.$$

(d) Relatively $\mathbf{P_0(P)}$-functions if for any $x \neq y$ in $\Re^n$,

$$\max_{i: x_i \neq y_i} [f(x) - f(y)]_i [g(x) - g(y)]_i \geq (>) 0.$$

(e) Relatively uniform $\mathbf{P}$-functions if there exists a constant $\eta > 0$ such that for any $x, y \in \Re^n$,

$$\max_{1 \leq i \leq n} [f(x) - f(y)]_i [g(x) - g(y)]_i \geq \eta \|x - y\|^2.$$

Note that relatively strongly monotone functions are relatively strictly monotone, and relatively strictly monotone functions are relatively monotone. Also we note that every relatively monotone (strictly monotone) functions are relatively $\mathbf{P}_0(\mathbf{P})$-functions.

The following Lemma (Tawhid 2008) is needed in our subsequent analysis.

Lemma 2.1 *Suppose that $f, g : \Re^n \to \Re^n$ and g is one-to-one and onto. Define $h : \Re^n \to \Re^n$ where $h := f \circ g^{-1}$. The following hold:*

Author Proof
uncorrected proof

Author Proof

(a) *f and g are relatively (strictly) monotone if and only if h is (strictly) monotone.*
(b) *If g is Lipschitz continuous, and f and g are relatively strongly monotone then h is strongly monotone.*
(c) *f and g are relatively $P_0(P)$-functions if and only if h is $P_0(P)$-function.*
(d) *If g is Lipschitz continuous, and f and g are relatively uniform P-functions, then h is uniform P-function.*

The following is a well-known result, see Harker and Pang (1990).

Proposition 2.1 *Let $f : R^n \to R^n$ and f is C^1 function,*

(a) *f is monotone if and only if $\nabla f(x)$ is a positive semi-definite Jacobian for all $x \in R^n$.*
(b) *f is strictly monotone if $\nabla f(x)$ is a positive definite Jacobian for all $x \in R^n$.*

Remark. Note that the converse of part (b) in Proposition 2.1 is not true in general.

3 Minimizing the merit function

Our objective in this article is to study GCP functions based on the NCP functions defined in Sect. 1.2. For given C^1- functions $f : R^n \to R^n$ and $g : R^n \to R^n$, we consider the associated GCP function Φ and the corresponding merit function

$$\Psi_*(\bar{x}) := \frac{1}{2}\|\Phi_*(\bar{x})\|^2 = \sum_{i=1}^{n} \psi_*(f_i(\bar{x}), g_i(\bar{x})), \tag{3.1}$$

where

$$\Phi_*(\bar{x}) := \begin{pmatrix} \phi_*(f_1(\bar{x}), g_1(\bar{x})) \\ \vdots \\ \phi_*(f_n(\bar{x}), g_n(\bar{x})) \end{pmatrix}, \tag{3.2}$$

and

$$\psi_*(a, b) := \frac{1}{2}\phi_*(a, b)^2, \tag{3.3}$$

with $* \in \{\{1, p\}, 1, 2, 3, 4, \{\theta, p\}\}$.

Now we let $\psi_{\alpha,\theta,p}(a, b) = \phi_{\alpha,\theta,p}(a, b)$ and denote the corresponding merit function as

$$\Psi_{\alpha,\theta,p}(x) := \sum_{i=1}^{n} \phi_{\alpha,\theta,p}(f_i(x), g_i(x)) = \sum_{i=1}^{n} \psi_{\alpha,\theta,p}(f_i(x), g_i(x)). \tag{3.4}$$

It should be recalled that

$$\Psi_*(\bar{x}) = 0 \Leftrightarrow \Phi_*(\bar{x}) = 0 \Leftrightarrow barx \text{ solves GCP}(f, g).$$

The authors in Gu and Tawhid (2014) used the concepts of relatively $\mathbf{P_0(P)}$-functions, relatively monotone, relatively strictly monotone in Definition 2.1 and the result in Lemma 2.1 to find the local/global minimum of Ψ_* (or a 'stationary point' of Ψ_*) which leads to a solution of the given generalized complementarity problem.

To weaken the hypotheses in the results in Gu and Tawhid (2014), we need to generalize the concept of a regular (strictly regular) point in Facchinei and Kanzow (1997), Ferris and Ralph (1995), Luca et al. (1996).

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

For given continuously differentiable functions f, g, and $x^* \in \Re^n$, we define the following index subsets of $I = \{1, 2, \ldots, n\}$:

$$\mathcal{C}(x^*) := \{i \in I : f_i(x^*) \geq 0, g_i(x^*) \geq 0, f_i(x^*)g_i(x^*) = 0\}, \quad \mathcal{R}(x^*) := I \backslash \mathcal{C}(x^*),$$
$$\mathcal{P}(x^*) := \{i \in \mathcal{R}(x^*) : f_i(x^*) > 0, g_i(x^*) > 0\}, \quad \mathcal{N}(x^*) := \mathcal{R}(x^*) \backslash \mathcal{P}(x^*).$$

Definition 3.1 Consider f, g, x^* and the index sets as above. Suppose that f and g are continuously differentiable and $\nabla g(x^*)$ is a nonsingular matrix. A vector $x^* \in \Re^n$ is called relatively regular (strictly relatively regular) with respect f and g if for every nonzero vector $z \in \Re^n$ such that

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0, \tag{3.5}$$

there exists a nonzero vector $s \in \Re^n$ such that

$$s_{\mathcal{C}} = 0, s_{\mathcal{P}} \geq 0, s_{\mathcal{N}} \leq 0, \quad \text{and} \quad s^T(\nabla g(x^*)^{-1} \nabla f(x^*))z \geq 0 (> 0). \tag{3.6}$$

Theorem 3.1 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Assume $\nabla g(x)$ is nonsingular for all $x \in R^n$. Suppose Φ_* is a GCP function of f and g satisfying the following conditions:*

$$\begin{aligned} i \in \mathcal{P} &\Rightarrow \Phi_i(x^*) > 0, \\ i \in \mathcal{N} &\Rightarrow \Phi_i(x^*) < 0, \\ i \in \mathcal{C} &\Rightarrow \Phi_i(x^*) = 0. \end{aligned} \tag{3.7}$$

Suppose $\Psi_* := \frac{1}{2}\|\Phi_*\|^2$ satisfies:

(i) *Ψ_* is continuously differentiable,*
(ii) *$(\nabla_a \psi_*(f_i(x), g_i(x))) > 0, (\nabla_b \psi_*(f_i(x), g_i(x))) > 0$, whenever $\Phi_{*i}(x) > 0$; and $(\nabla_a \psi_*(f_i(x), g_i(x))) < 0, (\nabla_b \psi_*(f_i(x), g_i(x))) < 0$, whenever $\Phi_{*i}(x) < 0$,*
(iii) *$\nabla_a \psi_*(f_i(x), g_i(x)) = \nabla_b \psi_*(f_i(x), g_i(x)) = 0$ whenever $\Phi_{*i}(x) = 0$.*

Suppose that x^* is a relatively regular point of f and g, then x^* is a stationary point of Ψ_* if and only if x^* is a solution of $GCP(f, g)$.

Proof "⇐" Suppose that x^* is a solution of $GCP(f, g)$, then $\Phi_*(x^*) = 0$, and from the property (iii), we have

$$\nabla \Psi_*(x^*) = \sum_{n=1}^{n} (\nabla f_i(x^*) \nabla_a \psi_*(f_i(x), g_i(x)) + \nabla g_i(x^*) \nabla_b \psi_*(f_i(x), g_i(x))) = 0,$$

that is, x^* is a stationary point of Ψ_*.

"⇒" Suppose that x^* is a stationary point of Ψ, i.e.,

$$\nabla \Psi_*(x^*) = \sum_{n=1}^{n} (\nabla f_i(x^*) \nabla_a \psi_*(f_i(x), g_i(x)) + \nabla g_i(x^*) \nabla_b \psi_*(f_i(x), g_i(x))) = 0$$

then by denoting

$$\nabla_a \psi_*(F(x^*), G(x^*)) = (\ldots, \nabla_a \psi_*(F_i(x^*), G_i(x^*)), \ldots)^T,$$

and similarly,

$$\nabla_b \psi_*(F(x^*), G(x^*)) = (\ldots, \nabla_b \psi_*(F_i(x^*), G_i(x^*)), \ldots)^T,$$

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

uncorrected proof

Author Proof

we have

$$\nabla F(x^*)\nabla_a \psi_*(F(^*, G(x^*)) + G(x^*)\nabla_b \psi_*(F(^*, G(x^*)) = 0. \tag{3.8}$$

Now multiply by $\nabla G(x^*)^{-1}$,

$$\nabla G(x^*)^{-1}\nabla F(x^*)\nabla_a \psi_*(F(x^*), G(x^*)) + \nabla_b \psi_*(F(x^*), G(x^*)) = 0. \tag{3.9}$$

Denote $z := \nabla_a \psi_*(F(x^*), G(x^*))$ and $y := \nabla_b \psi_*(F(x^*), G(x^*))$, then for any $s \in \Re^n$, we have

$$s^T \nabla G(x^*)^{-1}\nabla F(x^*)z + s^T y = 0. \tag{3.10}$$

We want to prove that x^* is a solution of $GCP(f, g)$, that is, $\Phi(x^*) = 0$. Suppose not, i.e., $\Phi(x^*) \neq 0$, then $\mathcal{R}(x^*) \neq \emptyset$ and $z_{\mathcal{C}} = 0$, $z_{\mathcal{P}} > 0$, $z_{\mathcal{N}} < 0$. Since x^* is a relatively regular point, the property (ii) holds, thus y and z have the same sign, by taking s satisfying (3.6), we have

$$s^T \nabla G(x^*)^{-1}\nabla F(x^*)z \geq 0 \quad \text{and} \quad s^T y > 0, \tag{3.11}$$

which contradicts (3.10). The proof is complete. □

Theorem 3.2 *Suppose $f : \Re^n \to \Re^n$ and $g : \Re^n \to \Re^n$ are continuously differentiable. Suppose Φ_* is a GCP function of f and g satisfying the following conditions:*

$$\begin{aligned} i \in \mathcal{P} &\Rightarrow \Phi_{*i}(\bar{x}) > 0,\\ i \in \mathcal{N} &\Rightarrow \Phi_{*i}(\bar{x}) < 0,\\ i \in \mathcal{C} &\Rightarrow \Phi_{*i}(\bar{x}) = 0. \end{aligned} \tag{3.12}$$

Suppose $\Psi_* := \frac{1}{2}\|\Phi_*\|^2$ satisfies:

(i) *Ψ_* is continuously differentiable,*
(ii) *$\nabla_a \psi_*(f_i(x), g_i(x)) > 0$, $\nabla_b \psi_*(f_i(x), g_i(x)) \geq 0$, whenever $\Phi_{*i}(x) > 0$; and $\nabla_a \psi_*(f_i(x), g_i(x)) < 0$, $\nabla_b \psi_*(f_i(x), g_i(x)) \leq 0$, whenever $\Phi_{*i}(x) < 0$,*
(iii) *$\nabla_a \psi_*(f_i(x), g_i(x)) = \nabla_b \psi_*(f_i(x), g_i(x)) = 0$ whenever $\Phi_{*i}(x) = 0$.*

Suppose that x^* is a strictly regular point of f and g, then x^* is a stationary point of Ψ_* if and only if x^* is a solution of $GCP(f, g)$.

Proof By a similar proof with Theorem 3.6, we can easily get the results. □

Remark 3.1 Since GCP functions in (1.5)–(1.11) satisfy the assumptions of Theorems 3.1 and 3.2, therefore the results of Theorems 3.1 and 3.2 are valid for these GCP functions, i.e., Theorem 3.1 and Theorem 3.2 are applicable to GCP functions in (1.5)–(1.11).

4 A descent direction algorithm

For the context nonlinear complementarity problem NCP, when f is C^1, Yamashita and Fukushima (1995), Geiger and Kanzow (1996), Chen and Pan (2008) proposed a descent method for minimizing the implicit Lagrangian function, square Fischer-Burmeister function and generalized Fischer-Burmeister, respectively, which does not require to compute the derivative of f and Ψ.

In this section, we present a descent algorithm for generalized complementarity problem based on the generalized Fischer-Burmeister function and its related merit function. In addition, we prove the global convergence of the algorithm. We assume that f and g are continuously differentiable and $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix.

uncorrected proof

Author Proof

Algorithm 4.1 Step 0 Given a GCP function Ψ_*, and $x_0 \in \Re^n$. Choose $\sigma \in (0,1)$ and $\beta \in (0,1)$. Set $k := 1$.

Step 1: If $\Psi_*(x^k) = 0$, then stop, otherwise go to Step 2.

Step 2: Consider the search direction as

$$d^k := -(\nabla g(x^k)^{-1})^T \nabla_a \psi_*(f(x^k), g(x^k)) \tag{4.1}$$

Step 3: Compute a step-size β^{m_k} where m_k is the smallest nonnegative integer m satisfying the Armijo-type condition:

$$\Psi_*(x^k + \beta^m d^k) \le (1 - \sigma\beta^{2m})\Psi_*(x^k). \tag{4.2}$$

Step 4: Set $x^{k+1} := x^k + \beta^{m_k} d^k$, $k := k+1$ and go to Step 1.

By the following Lemma, d^k is a descent direction of $\Psi_*(x^k)$ at x^k when $\nabla g(x)^{-1} \nabla f(x)$ is a positive semi-definite matrix.

Lemma 4.1 *Let f and g be continuously differentiable. Suppose that $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix and $\nabla g(x)^{-1} \nabla f(x)$ is a positive semi-definite matrix.*

Then as long as x^k is not a solution of the GCP, the direction defined as (4.1) satisfies the descent condition:

$$\nabla \Psi_*(x^k)^T d^k < 0.$$

Proof Assume that x^k not a solution of the GCP, then there exists a index subset $U \subseteq \{1, 2, \ldots, n\}$, such that $\Phi_{*i}(x^k) \neq 0$, $\forall i \in U$. By Proposition 3.1 in Gu and Tawhid (2014), we have $\nabla_a \psi_*(f_i(x), g_i(x)) \nabla_b \psi_*(f_i(x), g_i(x)) > 0$, $\forall i \in U$, then

$$\begin{aligned}
\nabla \Psi_*(x^k)^T d^k &= -(\nabla f(x^k) \nabla_a \psi_*(f(x^k), g(x^k)) \\
&\quad + \nabla g(x^k) \nabla_b \psi_*(f(x^k), g(x^k)))^T (\nabla g(x^k)^{-1})^T \nabla_a \psi_*(f(x^k), g(x^k)) \\
&= -\nabla_a \psi_*(f(x^k), g(x^k))^T [\nabla g(x^k)^{-1} \nabla f(x^k)]^T \nabla_a \psi_*(f(x^k), g(x^k)) \\
&\quad - \sum_{i=1}^{n} \nabla_a \psi_*(f_i(x), g_i(x)) \nabla_b \psi_*(f_i(x), g_i(x)) \\
&\le -\sum_{i \in U} \nabla_a \psi_*(f_i(x), g_i(x)) \nabla_b \psi_*(f_i(x), g_i(x)) \\
&< 0.
\end{aligned} \tag{4.3}$$

Note that we get the first inequality because $\nabla g(x)^{-1} \nabla f(x)$ is a positive semi-definite matrix. The proof is complete. □

Remark 4.1 It is known that if the map is monotone, its Jacobian is positive semi-definite (see, e.g., Ortega and Rheinboldt 1970, p. 142). In view of Lemma 2.1 and Proposition 2.1, we get the following Corollary, d^k is a descent direction of $\Psi_*(x^k)$ at x^k under monotonicity assumptions.

In view of Part (a) in Lemma 2.1, Part (a) in Proposition 2.1 and Lemma 4.1, we have the following result.

Corollary 4.1 *Let f and g be continuously differentiable. Suppose that $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix. Assume f and g are relatively monotone.*

Then as long as x^k is not a solution of the GCP, the direction defined as 4.1 satisfies the descent condition:

$$\nabla \Psi_*(x^k)^T d^k < 0.$$

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Lemma 4.2 *Let f and g be continuously differentiable. Suppose that $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix and $\nabla g(x)^{-1}\nabla f(x)$ is a positive semi-definite matrix. Then Step 3 is well defined.*

Proof It is sufficient to show that there exists a nonnegative integer m_k in Step 3 of Algorithm 4.1 whenever x^k is not a solution. Assume that the conclusion does not hold. Then for any $m > 0$,

$$\Psi_*(x^k + \beta^m d^k) - \Psi_*(x^k) > -\sigma\beta^{2m}\Psi_*(x^k).$$

Dividing by β^m on two sides and taking $m \to +\infty$, then we have

$$\langle \nabla\Psi_*(x^k), d^k\rangle \geq 0.$$

This contradicts Lemma 4.1. Hence, we can find an integer m_k in Step 3. □

From Lemmas 4.1 and 4.2, we have Algorithm 4.1 is well defined. By Remark 4.1, we have the following.

Corollary 4.2 *Let f and g be continuously differentiable. Suppose that $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix. Assume f and g are relatively monotone. Then Step 3 is well defined.*

The next Proposition is a global convergence result for Algorithm 4.1.

Proposition 4.1 *Let f and g be continuously differentiable. Assume the assumptions of Theorem 3.1 are satisfied. Suppose that $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix and $\nabla g(x)^{-1}\nabla f(x)$ is a positive semi-definite matrix. Further assume that the level set $\mathcal{L}(\Psi,\gamma) := \{x \in \Re^n : \Psi_*(x) \leq \gamma\}$ is bounded for any γ. Then the sequence $\{x^k\}$ generated by Algorithm 4.1 has at least one accumulation point and any accumulation point is a solution of the GCP.*

Proof We first show that the sequence $\{x^k\}$ generated by Algorithm 4.1 has at least one accumulation point. By the descent property of Algorithm 4.1, the sequence $\{\Psi(x^k)\}$ is decreasing. Since the level set $\mathcal{L}(\Psi_*, \Psi_*(x^0))$ is bounded, then we have the sequence $\{x^k\}$ is bounded. Thus $\{x^k\}$ has at least one accumulation point.

Next, we prove that every accumulation point is a solution of the GCP. Let x^* be an arbitrary accumulation point of the generated sequence $\{x^k\}$. Then there exists a subsequence, for simplicity, denoted by $\{x^k\}$ which converges to x^*. Since Ψ_* and g are continuously differentiable, $\nabla\Psi_*(\cdot)$ and $\nabla g(\cdot)$ are continuous. Thus

$d^k \to d^*$ as $k \to \infty$. Now in our proof, we will consider two cases:

Case (a): If there exists a constant $\bar\beta$ such that $\beta^{m^k} \geq \bar\beta > 0$ for all $k \in \{1, 2, \ldots\}$. Then, from step 3, we have for all $k \in \{1, 2, \ldots\}$,

$$\begin{aligned} 0 \leq \Psi_*(x^{k+1}) &\leq (1-\sigma\bar\beta^2)\Psi_*(x^k) \\ &\leq (1-\sigma\bar\beta^2)^k\Psi_*(x^0) \\ &\to 0, \end{aligned}$$

by $\sigma \in (0, 1)$ and $\bar\beta \in (0, 1)$. Thus, we have $\Psi_*(x^*) = 0$ which implies x^* is a solution of GCP.

Case (b): We consider the other case where there exists a further subsequence such that $\beta^{m^k} \to 0$. From Step 3, we have

$$\Psi_*(x^k + \beta^{m_k-1}d^k) - \Psi_*(x^k) > -\sigma\beta^{2m_k-1}\Psi_*(x^k).$$

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Dividing both sides by β^{m_k-1} and passing to the limit on the subsequence, we get

$$\langle \nabla \Psi_*(x^*), d^* \rangle \geq 0,$$

which implies x^* is a solution of GCP. □

In view of Remark 4.1, we have the following.

Corollary 4.3 *Let f and g be continuously differentiable. Suppose that $\forall x \in \Re^n$, $\nabla g(x)$ is a nonsingular matrix. Assume f and g are relatively monotone. Further assume that the level set $\mathcal{L}(\Psi_*, \gamma) := \{x \in \Re^n : \Psi_*(x) \leq \gamma\}$ is bounded for any γ. Then the sequence $\{x^k\}$ generated by Algorithm 4.1 has at least one accumulation point and any accumulation point is a solution of the GCP.*

5 Numerical experiments

In the following, we implement Algorithm 4.1 for GCP(f, g) where f and g are continuously differentiable. All numerical experiments are done by a Windows PC using MatLab with CPU of 1.90 GHz and RAM of 8.00 GB. The values of σ and β were set to 1.0×10^{-10} and 0.2, respectively. These settings were found to work well on average across the different test problems. We terminate Algorithm 4.1 if one of the following conditions is satisfied:

1. $\Psi_*(x^k) \leq 10^{-9}$ and $d^k \leq 10^{-3}$;
2. the steplength is less than 10^{-9};
3. the number of iterations is more than 100,000.

[3] To test the effectiveness of the to test the descent direction algorithm, 4 test problems were used. Each of the 7 types of GCP functions was used as $\Psi_*(x)$ with several different values of p, α, and θ. In the resulting tables we show results for $p \in \{1.5, 2, 3\}$, $\alpha \in \{0.01, 0.1, 1, 10\}$, and $\theta \in \{0.25, 0.5, 0.75\}$.

Test Problem 1: (*Implicit complementarity problems*) (Outrata and Zowe 1995)
We define this problems as follows: Find $x^* \in \Re^5$ such that

$$f(x^*) = x^* - m(x^*) \geq 0, \quad g(x^*) \geq 0, \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle = 0,$$

where

$$g(x) := \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and $m(x) = \pi(g(x)) : \Re^n \rightarrow \Re^n$ is twice continuously differentiable. The followings are two choices of function $\pi(\cdot)$:

(a) Linear case: $\pi_i(t) = -0.5 - t_i$, $i = 1, 2, 3, 4$;
(b) Non-Linear case: $\pi_i(t) = -0.5 - 1.5t_i + 0.25t_i^2$, $i = 1, 2, 3, 4$.

We implement Algorithm 4.1 using the following three initial (starting) points:

(a) $(0.0, 0.0, 0.0, 0.0)^T$,
(b) $(-0.5, -0.5, -0.5, -0.5)^T$,
(c) $(-1.0, -1.0, -1.0, -1.0)^T$.

uncorrected proof

Author Proof

Table 1 Numerical results for GCP function ϕ_1 for the linear case of test problem 1

Type	ST	α	$p = 1.5$		$p = 2$		$p = 3$	
			RES	IT	RES	IT	RES	IT
ϕ_1	(a)	0.01	7.05E−08	18	1.08E−07	13	1.21E−07	14
	(a)	0.1	4.81E−09	13	6.63E−08	14	4.22E−08	16
	(a)	1	2.47E−04	3	4.67E−03	3	2.96E−03	4
	(a)	10	1.96E−08	16	1.67E−08	15	1.19E−08	13
	(b)	0.01	2.64E−07	10	5.21E−08	15	1.26E−07	11
	(b)	0.1	4.30E−08	13	2.20E−07	10	6.04E−08	8
	(b)	1	5.26E−06	9	2.67E−05	14	2.64E−07	20
	(b)	10	1.95E−07	26	2.01E−07	88	1.94E−07	75
	(c)	0.01	5.58E−08	14	9.12E−08	10	2.21E−07	15
	(c)	0.1	2.40E−01	2	3.50E−07	17	5.39E−08	15
	(c)	1	2.70E−05	10	1.46E−05	17	5.56E−08	20
	(c)	10	1.86E−07	49	2.04E−07	103	1.22E−07	32

Table 2 Numerical results for GCP function ϕ_1 for the nonlinear case of test problem 1

Type	ST	α	$p = 1.5$		$p = 2$		$p = 3$	
			RES	IT	RES	IT	RES	IT
ϕ_1	(a)	0.01	7.71E−08	23	6.31E−07	17	1.30E−07	19
	(a)	0.1	3.35E−07	17	5.49E−07	18	2.94E−07	18
	(a)	1	3.28E−04	3	5.69E−03	4	2.87E−08	27
	(a)	10	5.97E−09	15	4.40E−08	16	2.03E−09	17
	(b)	0.01	6.86E−08	16	7.60E−08	15	2.75E−07	14
	(b)	0.1	8.98E−08	8	4.17E−07	9	2.61E−07	10
	(b)	1	5.06E−07	11	3.29E−04	8	1.73E−02	5
	(b)	10	1.79E−07	18	8.40E−08	35	1.12E−07	28
	(c)	0.01	6.75E−07	14	6.48E−08	10	3.25E−07	16
	(c)	0.1	2.37E−01	2	1.26E−07	15	2.02E−07	15
	(c)	1	5.06E−07	9	2.70E−07	26	2.98E−07	80
	(c)	10	5.38E−08	43	3.97E−08	12	1.18E−08	13

Test Problem 2: (*Nash equilibrium problem*)

The Nash equilibrium problem is a part of the MCPLIB library of problems (Dirkse and Ferris 1994). Here $n = 10$. The function f is a **P**-function on the strictly positive orthant, but not twice differentiable.

$$F(x) = \nabla C(x) - p(\xi) - \nabla p(\xi)x,$$
$$G(x) = x.$$

For details on the functions $C(x)$, $p(\xi)$ please refer to the MCPLIB problem set (Dirkse and Ferris 1994) which is publicly available.

Uncorrected proof

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

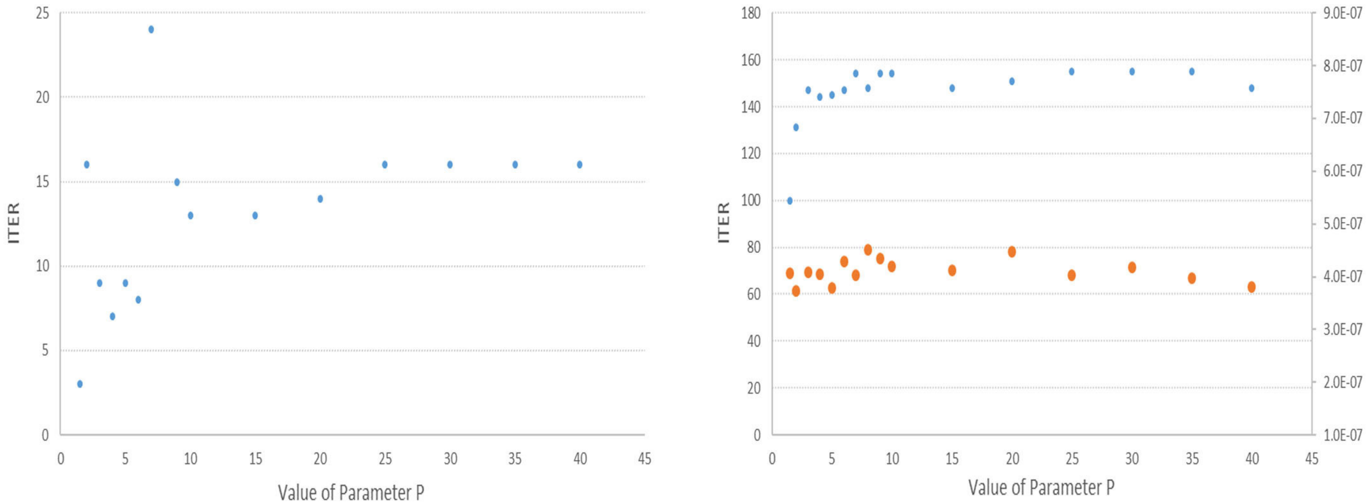

Fig. 1 The *blue markers* on the graphs indicate the number of iterations to reach the stopping limit with different values of P with test problem 1b based on ϕ_1, $\alpha = 1$ starting point (a) (*left*) and test problem 2, ϕ_3, $\alpha = 0.1$ and starting point (a) (*right*). In addition, *red markers* to indicate the value of RES for different values of P is also shown for test problem 2 (*right*)

Journal: **40314** Article No.: **0328** TYPESET DISK LE CP Disp.:**2016/2/26** Pages: **26** Layout: **Small**

Author Proof

Table 3 Average number of iterations to reach solution for GCP functions in test problem 1

GCP Function type	Example 1 (linear) ITER	Example 1 (non-linear) ITER
1	18.11	15.78
2	20.41	17.39
3	23.11	16.89
4	**12.36**	**13.06**
5	**12.36**	**13.06**
6	15.93	16.78
7	16.29	18.13

uncorrected proof

In this example we use the following two starting points:

(a) $(0.0, \ldots, 0.0)^T$,
(b) $(1, \ldots, 1)^T$.

Test Problem 3: (*Kojima-Shindo problem*) (Dirkse and Ferris 1994)
Here $n = 4$ and the function F is not a P_0-function such that

$$F(x) := \begin{bmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_2^2 + x_1 + 10x_3 + 2x_4 - 2 \\ 3x_1 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3 \end{bmatrix}$$

and

$$G(x) := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T.$$

Two starting points are used: 4

(a) $(0.0, \ldots, 0.0)^T$,
(b) $(1, \ldots, 1)^T$,

Test Problem 4: (*Mathiesen problem*) (Dirkse and Ferris 1994)
We consider Mathiesen's small example of a Walrasian equilibrium model where $n = 4$, and

$$F(x) = \begin{bmatrix} -x_2 + x_3 + x_4 \\ x_1 - \alpha(b_2x_3 + b_3x_4)/x_2 \\ b_2 - x_1 - (1-\alpha)(b_2x_3 + b_3x_4)/x_3 \\ b_3 - x_1 \end{bmatrix},$$

and

$$G(x) := \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T.$$

Two starting points are used:

(a) $(0.0, \ldots, 0.0)^T$,
(b) $(1, \ldots, 1)^T$,

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Table 4 Numerical results for Algorithm 4.1 based on the GCP functions for the linear case of test problem 1

Type	ST	θ	α	$p = 1.5$		$p = 2$		$p = 3$	
				RES	IT	RES	IT	RES	IT
ϕ_p	(a)	–	–	7.0E−10	24	1.82E−10	17	3.32E−10	20
	(b)	–	–	2.99E−10	16	1.83E−10	17	9.48E−10	15
	(c)	–	–	4.16E−10	19	9.32E−10	15	6.23E−10	20
ϕ_1	(a)	–	0.01	7.05E−08	18	1.08E−07	13	1.21E−07	14
	(a)	–	0.1	4.81E−09	13	6.63E−08	14	4.22E−08	16
	(a)	–	1	2.47E−04	3	4.67E−03	3	2.96E−03	4
	(a)	–	10	1.96E−08	16	1.67E−08	15	1.19E−08	13
	(b)	–	0.01	2.64E−07	10	5.21E−08	15	1.26E−07	11
	(b)	–	0.1	4.30E−08	13	2.20E−07	10	6.04E−08	8
	(b)	–	1	5.26E−06	9	2.67E−05	14	2.64E−07	20
	(b)	–	10	1.95E−07	26	2.01E−07	88	1.94E−07	75
	(c)	–	0.01	5.58E−08	14	9.12E−08	10	2.21E−07	15
	(c)	–	0.1	2.40E−01	2	3.50E−07	17	5.39E−08	15
	(c)	–	1	2.70E−05	10	1.46E−05	17	5.56E−08	20
	(c)	–	10	1.86E−07	49	2.04E−07	103	1.22E−07	32
ϕ_2	(a)	–	0.01	7.05E−08	18	1.08E−07	13	1.21E−07	14
	(a)	–	0.1	4.81E−09	13	6.63E−08	14	4.22E−08	16
	(a)	–	1	2.47E−04	3	4.67E−03	3	2.96E−03	4
	(a)	–	10	1.88E−07	105	1.16E−07	17	3.38E−09	19
	(b)	–	0.01	2.64E−07	10	5.21E−08	15	1.26E−07	11
	(b)	–	0.1	4.30E−08	13	2.20E−07	10	6.04E−08	8
	(b)	–	1	5.26E−06	9	2.67E−05	14	2.64E−07	20
	(b)	–	10	1.95E−07	26	2.01E−07	88	1.94E−07	75
	(c)	–	0.01	5.58E−08	14	9.12E−08	10	2.21E−07	15
	(c)	–	0.1	2.40E−01	2	3.50E−07	17	5.39E−08	15
	(c)	–	1	2.70E−05	10	1.46E−05	7	5.56E−08	20
	(c)	–	10	1.86E−07	49	2.04E−07	103	1.22E−07	32
ϕ_3	(a)	–	0.01	2.47E−08	16	1.52E−07	13	4.07E−07	13
	(a)	–	0.1	2.86E−07	13	5.80E−08	15	8.17E−08	14
	(a)	–	1	1.51E−07	12	6.28E−07	12	1.16E−07	16
	(a)	–	10	9.54E−08	14	1.17E−07	14	7.77E−08	14
	(b)	–	0.01	4.15E−07	9	4.77E−08	15	2.19E−07	11
	(b)	–	0.1	1.99E−07	9	6.86E−08	11	1.79E−09	7
	(b)	–	1	6.11E−07	10	1.39E−07	12	4.39E−07	10
	(b)	–	10	7.62E−08	12	1.38E−07	11	1.04E−07	12
	(c)	–	0.01	1.55E−07	14	3.12E−07	11	2.58E−07	15
	(c)	–	0.1	1.99E−07	11	6.67E−08	13	9.64E−08	15
	(c)	–	1	6.87E−08	14	2.3E−07	12	7.79E−07	10
	(c)	–	10	6.10E−08	12	3.35E−08	10	6.32E−08	13

uncorrected proof

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Table 5 Numerical results for Algorithm 4.1 based on the GCP functions for the linear case of test problem 1

Type	ST	θ	α	$p = 1.5$		$p = 2$		$p = 3$	
				RES	IT	RES	IT	RES	IT
ϕ_4	(a)	–	0.01	2.74E−08	16	1.52E−07	13	4.07E−07	13
	(a)	–	0.1	2.86E−07	13	5.80E−08	15	8.17E−08	14
	(a)	–	1	1.51E−07	12	6.28E−07	12	1.16E−07	16
	(a)	–	10	9.54E−08	14	1.17E−07	14	7.77E−08	14
	(b)	–	0.01	4.15E−07	9	4.77E−08	15	2.19E−07	11
	(b)	–	0.1	1.99E−07	9	6.86E−08	11	1.79E−09	7
	(b)	–	1	6.11E−07	10	1.39E−07	12	4.39E−07	10
	(b)	–	10	7.62E−08	12	1.38E−07	11	1.04E−07	12
	(c)	–	0.01	1.55E−07	14	3.12E−07	11	2.58E−07	15
	(c)	–	0.1	1.99E−07	11	6.67E−08	13	9.64E−08	15
	(c)	–	1	6.87E−08	14	2.3E−07	12	7.79E−07	10
	(c)	–	10	6.10E−08	12	3.35E−08	10	6.32E−08	13
$\phi_{\theta,p}$	(a)	0.25	–	7.53E−09	21	3.47E−08	19	6.92E−09	21
	(a)	0.5	–	1.06E−08	19	8.36E−09	16	5.27E−09	15
	(a)	0.75	–	2.66E−07	16	4.29E−07	13	2.69E−07	13
	(b)	0.25	–	9.11E−09	19	1.23E−08	20	4.47E−09	20
	(b)	0.5	–	1.07E−08	16	7.14E−09	18	9.84E−09	18
	(b)	0.75	–	4.66E−07	10	3.06E−07	9	7.09E−08	6
	(c)	0.25	–	5.35E−09	19	8.60E−09	19	1.27E−08	19
	(c)	0.5	–	6.62E−09	16	6.34E−09	17	7.23E−09	17
	(c)	0.75	–	5.32E−07	9	3.81E−07	13	4.95E−07	12

Tables 1 and 2 show the results of our numerical tests for the test problems 1a and 1b. In these tables, the first column lists GCP functions mentioned in Examples 1–7, the second column shows the staring points, and the third column indicates various values for α. RES indicates the value of the merit function, and the number of iterations is shown in IT. We used the algorithm to solve 5 different test problems in total including 2 instances of test problem 1.

The GCP functions include a few parameters which can be set. These parameters are p, α, and θ. We tested several different combinations within the range of permissible values for each parameter. In our numerical results there was a general trend that decreasing p leads to faster convergence. This can be seen both versions of test problem 1, test problem 3 and test problem 4. Examples of iteration count when p is increased can be found in Fig. 1. The value of p can be any value $\in (1, \infty)$ and although not every test cases showed quick convergence to the optimal solution, our numerical tests indicate increasing p does not improve performance. In Fig. 1 on the right the RES is also shown and it was found that the value of p did not greatly impact the quality of the final solution found in terms RES. For this reason we chose to use $p \in \{1.5, 2, 3\}$ as our initial test parameters in our algorithm. For the parameter α, there was no clear trend for which value tended to provide the best solution. Different combinations of α in conjunction with different GCP functions can improve the final result of the algorithm. For this reason the range $\alpha \in \{0.01, 0.1, 1, 10\}$ is used.

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Table 6 Numerical results for Algorithm 4.1 based on the GCP functions for the linear case of test problem 1

Type	ST	θ	α	$p = 1.5$		$p = 2$		$p = 3$	
				RES	IT	RES	IT	RES	IT
$\phi_{\alpha,\theta,p}$	(a)	0.25	0.01	9.01E−09	21	3.24E−09	19	7.61E−09	21
	(a)	0.5	0.01	1.28E−08	17	7.22E−09	14	5.91E−09	16
	(a)	0.75	0.01	3.24E−07	14	6.23E−07	14	2.46E−07	14
	(a)	0.25	0.1	1.77E−08	19	1.08E−08	21	1.46E−08	22
	(a)	0.5	0.1	8.27E−09	18	6.03E−09	19	6.73E−09	18
	(a)	0.75	0.1	6.47E−07	12	3.68E−08	8	3.65E−07	12
	(a)	0.25	1	3.04E−09	21	7.01E−09	19	1.90E−08	16
	(a)	0.5	1	1.15E−08	30	1.37E−08	27	1.50E−08	25
	(a)	0.75	1	6.19E−08	9	7.51E−08	13	1.89E−07	13
	(a)	0.25	10	2.99E−10	14	6.48E−11	14	8.49E−10	13
	(a)	0.5	10	1.89E−08	20	5.30E−09	21	6.74E−09	21
	(a)	0.75	10	2.10E−09	16	2.49E−09	15	3.88E−09	14
	(b)	0.25	0.01	8.82E−09	19	1.25E−08	20	4.93E−09	20
	(b)	0.5	0.01	1.25E−08	16	8.33E−09	18	1.15E−08	18
	(b)	0.75	0.01	3.12E−07	10	6.43E−07	8	3.38E−08	7
	(b)	0.25	0.1	6.44E−09	19	4.41E−09	20	2.55E−08	19
	(b)	0.5	0.1	5.66E−09	18	7.70E−09	19	1.02E−08	19
	(b)	0.75	0.1	4.29E−09	7	4.14E−07	8	3.63E−07	11
	(b)	0.25	1	1.15E−08	18	1.58E−08	18	2.19E−08	17
	(b)	0.5	1	1.93E−08	24	1.74E−08	24	1.76E−08	24
	(b)	0.75	1	2.07E−07	10	2.11E−07	9	4.98E−07	8
	(b)	0.25	10	4.55E−09	13	5.73E−11	13	4.90E−10	13
	(b)	0.5	10	4.55E−09	19	7.35E−09	19	8.49E−09	19
	(b)	0.75	10	1.76E−09	14	1.95E−09	13	2.12E−09	12
	(c)	0.25	0.01	5.20E−09	19	8.36E−09	19	1.23E−08	19
	(c)	0.5	0.01	6.74E−09	16	6.33E−09	17	7.06E−09	17
	(c)	0.75	0.01	4.30E−07	14	3.53E−07	13	4.67E−07	12
	(c)	0.25	0.1	3.83E−09	19	6.56E−09	19	9.67E−09	19
	(c)	0.5	0.1	5.53E−09	16	8.51E−09	16	6.96E−09	16
	(c)	0.75	0.1	5.11E−07	13	5.88E−07	12	2.86E−07	12
	(c)	0.25	1	6.33E−09	16	7.31E−09	15	1.95E−08	13
	(c)	0.5	1	1.52E−08	30	1.72E−08	30	1.82E−08	30
	(c)	0.75	1	3.34E−07	9	1.26E−07	10	6.28E−08	10
	(c)	0.25	10	3.16E−10	12	3.00E−09	9	1.85E−09	9
	(c)	0.5	10	2.04E−08	18	2.62E−09	21	4.71E−09	18
	(c)	0.75	10	9.39E−10	14	3.78E−09	12	2.62E−09	12

In this paper, 7 different GCP functions were tested. It is of interest to see which GCP function type performs best. In Table 3 the average number of iterations taken to reach the stopping criteria for the test problems 1a, 1b across the range of different values of p, α, and θ is given. The GCP functions corresponding to type 4 and 5 were lowest which indicates faster

Uncorrected proof

Table 7 Numerical results for Algorithm 4.1 based on the GCP functions for the nonlinear case of test problem 1

Type	ST	θ	α	$p=1.5$		$p=2$		$p=3$	
				RES	IT	RES	IT	RES	IT
ϕ_p	(a)	–	–	6.23E−07	19	2.13E−07	17	2.98E−07	19
	(b)	–	–	8.25E−08	15	5.79E−07	14	5.40E−07	14
	(c)	–	–	3.11E−07	16	1.49E−07	13	5.55E−07	15
ϕ_1	(a)	–	0.01	7.17E−08	23	6.31E−07	17	1.30E−07	19
	(a)	–	0.1	3.35E−07	17	5.49E−07	18	2.94E−07	18
	(a)	–	1	3.28E−04	3	5.69E−03	4	2.87E−08	27
	(a)	–	10	5.97E−09	15	4.40E−08	16	2.03E−09	17
	(b)	–	0.01	6.86E−08	16	7.60E−08	15	2.75E−07	14
	(b)	–	0.1	8.98E−08	8	4.17E−07	9	2.61E−07	10
	(b)	–	1	5.06E−07	11	3.29E−04	8	1.73E−02	5
	(b)	–	10	1.79E−07	18	8.40E−08	35	1.12E−07	28
	(c)	–	0.01	6.75E−07	14	6.48E−08	10	3.25E−07	16
	(c)	–	0.1	2.37E−01	2	1.26E−07	15	2.02E−07	15
	(c)	–	1	5.06E−07	9	2.70E−07	26	2.98E−07	80
	(c)	–	10	5.38E−08	43	3.97E−08	12	1.18E−08	13
ϕ_2	(a)	–	0.01	7.17E−08	23	6.31E−07	17	1.30E−07	19
	(a)	–	0.1	3.35E−07	17	5.49E−08	18	2.94E−07	18
	(a)	–	1	3.28E−04	3	5.69E−03	4	2.87E−08	27
	(a)	–	10	6.79E−08	25	5.25E−07	14	5.08E−07	23
	(b)	–	0.01	6.86E−08	16	7.60E−08	15	2.75E−07	14
	(b)	–	0.1	8.98E−08	8	4.17E−07	9	2.61E−07	10
	(b)	–	1	5.06E−07	11	3.29E−05	8	1.73E−02	5
	(b)	–	10	5.45E−09	16	6.03E−07	19	3.54E−08	14
	(c)	–	0.01	6.75E−07	14	6.48E−08	10	3.25E−07	16
	(c)	–	0.1	2.37E−01	2	1.26E−07	15	2.02E−07	15
	(c)	–	1	5.06E−07	9	2.70E−05	26	2.98E−07	80
	(c)	–	10	5.38E−08	43	3.97E−07	12	1.18E−08	13
ϕ_3	(a)	–	0.01	1.55E−08	19	4.70E−07	17	9.94E−07	16
	(a)	–	0.1	1.11E−08	14	3.88E−07	12	1.18E−06	15
	(a)	–	1	7.21E−07	14	1.56E−06	13	7.59E−07	20
	(a)	–	10	5.70E−08	14	4.06E−08	14	7.28E−08	13
	(b)	–	0.01	3.24E−07	15	6.54E−07	13	3.00E−07	14
	(b)	–	0.1	3.89E−07	11	9.37E−07	11	4.85E−07	12
	(b)	–	1	1.95E−07	10	1.15E−06	16	4.50E−07	17
	(b)	–	10	6.48E−08	13	3.50E−08	12	1.12E−07	11
	(c)	–	0.01	2.05E−08	13	2.58E−07	10	2.82E−07	16
	(c)	–	0.1	4.61E−07	11	7.10E−07	15	4.38E−07	10
	(c)	–	1	2.10E−07	9	7.13E−07	8	8.33E−07	6
	(c)	–	10	6.94E−08	13	7.92E−09	11	4.45E−07	12

Author Proof
uncorrected proof

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Table 8 Numerical results for Algorithm 4.1 based on the GCP functions for the nonlinear case of test problem 1

Type	ST	θ	α	$p = 1.5$		$p = 2$		$p = 3$	
				RES	IT	RES	IT	RES	IT
ϕ_4	(a)	–	0.01	1.55E−08	19	4.70E−07	17	9.94E−07	16
	(a)	–	0.1	1.11E−08	14	3.88E−07	12	1.18E−06	15
	(a)	–	1	7.21E−07	14	1.56E−06	13	7.59E−07	20
	(a)	–	10	5.70E−08	14	4.06E−08	14	7.28E−08	13
	(b)	–	0.01	3.24E−07	15	6.54E−07	13	3.00E−07	14
	(b)	–	0.1	3.89E−07	11	9.37E−07	11	4.58E−07	12
	(b)	–	1	1.95E−07	10	1.15E−06	16	4.50E−07	17
	(b)	–	10	6.48E−08	13	3.50E−08	12	1.12E−07	11
	(c)	–	0.01	2.05E−08	13	2.58E−07	10	2.82E−07	16
	(c)	–	0.1	4.61E−07	11	7.10E−07	15	4.38E−07	10
	(c)	–	1	2.10E−07	9	7.13E−07	8	8.33E−07	6
	(c)	–	10	6.94E−08	13	7.92E−09	11	4.45E−07	12
$\phi_{\theta,p}$	(a)	0.25	–	4.24E−09	19	6.27E−09	20	3.51E−09	20
	(a)	0.5	–	1.33E−08	25	1.55E−08	26	1.52E−08	22
	(a)	0.75	–	2.73E−07	15	2.24E−07	13	4.47E−07	10
	(b)	0.25	–	4.44E−09	15	5.47E−09	18	6.74E−09	18
	(b)	0.5	–	1.49E−08	16	1.28E−08	24	1.61E−08	24
	(b)	0.75	–	1.81E−07	10	2.27E−07	9	1.49E−07	8
	(c)	0.25	–	3.43E−09	17	7.33E−09	17	4.69E−09	17
	(c)	0.5	–	1.34E−08	21	1.29E−08	17	1.12E−08	22
	(c)	0.75	–	2.94E−07	11	1.88E−07	8	2.68E−07	11

convergence. This suggests without choosing a specific set of values for p, α and θ these 2 GCP functions would on average give a reasonable final solution. The numerical results [6] also show that the range of results can vary greatly depending on the selected problem, GCP function and parameters of the algorithm. Choosing specific ϕ and optimizing the value of p, α, and θ for a specific problem can improve performance. For example, from Tables 1 and 2, we could see that ϕ_1 is best with $p = 1.5$ and $\alpha = 1$. The results for ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are quite similar from the results shown in Tables 4, 5, 6, 7 and 8. Also from Tables 5 and 8, when $\theta = 0.75$ and $p = 1.5$, $\phi_{\theta,p}$ performs particularly well for those examples. Finally, $\phi_{\alpha,\theta,p}$ appears to work best for test problems 1a and 1b when $\theta = 0.75$, and $p = 1.5$ (Table 9).

Test problems 1a and 1b were solvable in all test cases within our specified stopping criteria and a good approximation to the optimal solution was reached. For test problems 2, 3 and 4 there was a wider range of convergence behavior and in some cases the algorithm failed to converge to the optimal solution. Figures 2 and 3 show some examples of parameter settings for the different examples where convergence was rapid. The convergence behaviors in Figs. 2, 3 suggest that the algorithm may linear convergence. Similar behavior was also observed in other test cases. How to find the optimal parameters for individual problems is a point that merits further research. The starting point is an important determinant for the rate of convergence of the algorithm. The starting points for the problems were taken from the problem references where available.

Uncorrected proof

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

Table 9 Numerical results for Algorithm 4.1 based on the GCP functions for the nonlinear case of test problem 1

Type	ST	θ	α	$p = 1.5$		$p = 2$		$p = 3$	
				RES	IT	RES	IT	RES	IT
$\phi_{\alpha,\theta,p}$	(a)	0.25	0.01	2.11E−08	19	2.18E−09	21	2.03E−08	19
	(a)	0.5	0.01	1.18E−08	21	1.45E−08	28	1.71E−08	21
	(a)	0.75	0.01	4.94E−07	15	3.01E−07	11	1.90E−07	10
	(a)	0.25	0.1	3.16E−09	20	4.40E−09	16	2.42E−08	16
	(a)	0.5	0.1	1.40E−08	25	1.68E−08	22	1.32E−08	21
	(a)	0.75	0.1	2.91E−07	13	2.15E−07	11	4.50E−08	11
	(a)	0.25	1	5.30E−09	18	6.60E−09	17	4.16E−09	17
	(a)	0.5	1	1.55E−08	45	1.91E−08	43	2.29E−08	42
	(a)	0.75	1	1.78E−08	12	2.63E−07	12	3.66E−07	11
	(a)	0.25	10	5.42E−09	14	2.28E−09	13	2.77E−09	13
	(a)	0.5	10	4.46E−10	20	2.16E−09	19	2.88E−09	19
	(a)	0.75	10	1.83E−08	16	3.63E−09	15	9.07E−09	15
	(b)	0.25	0.01	1.03E−08	15	5.19E−09	18	5.57E−09	18
	(b)	0.5	0.01	9.59E−09	20	1.03E−08	25	1.62E−08	22
	(b)	0.75	0.01	1.35E−07	11	1.24E−07	9	3.20E−07	7
	(b)	0.25	0.1	2.36E−09	19	6.24E−09	17	5.43E−09	18
	(b)	0.5	0.1	1.02E−08	24	1.32E−08	23	1.60E−08	28
	(b)	0.75	0.1	1.95E−07	9	3.58E−07	7	4.31E−07	7
	(b)	0.25	1	1.37E−09	18	3.11E−09	16	1.62E−09	15
	(b)	0.5	1	1.59E−08	45	1.76E−08	46	1.80E−08	46
	(b)	0.75	1	1.05E−07	11	2.73E−07	9	1.50E−07	9
	(b)	0.25	10	1.46E−07	9	2.39E−09	15	3.93E−08	13
	(b)	0.5	10	3.60E−08	17	1.19E−09	17	1.38E−09	17
	(b)	0.75	10	3.61E−09	16	5.89E−09	13	8.44E−09	13
	(c)	0.25	0.01	4.22E−08	15	7.12E−09	17	4.06E−09	17
	(c)	0.5	0.01	1.64E−08	21	1.58E−08	8	1.28E−08	21
	(c)	0.75	0.01	1.90E−07	11	6.56E−08	7	3.87E−07	11
	(c)	0.25	0.1	3.21E−09	17	4.86E−09	17	2.43E−09	15
	(c)	0.5	0.1	1.04E−08	24	1.06E−08	25	1.36E−08	24
	(c)	0.75	0.1	1.57E−07	10	2.58E−07	11	2.01E−07	11
	(c)	0.25	1	3.51E−09	16	2.93E−09	15	2.95E−08	16
	(c)	0.5	1	1.47E−08	44	2.34E−08	46	1.56E−08	47
	(c)	0.75	1	2.90E−08	7	9.41E−10	9	6.32E−09	6
	(c)	0.25	10	1.80E−18	18	1.41E−08	18	3.26E−08	17
	(c)	0.5	10	1.52E−09	19	2.03E−08	17	1.98E−08	17
	(c)	0.75	10	1.83E−09	15	5.45E−09	13	9.56E−10	13

uncorrected proof

The GCP functions allow for the reformulation of the GCP into a global minimization problem. Therefore, it should be possible to make use of existing global optimization algorithms to solve the merit functions. How the parameters p, α and θ affect the results of other algorithms is something that can be explored in future works.

Journal: 40314 Article No.: 0328 TYPESET DISK LE CP Disp.:2016/2/26 Pages: 26 Layout: Small

Author Proof

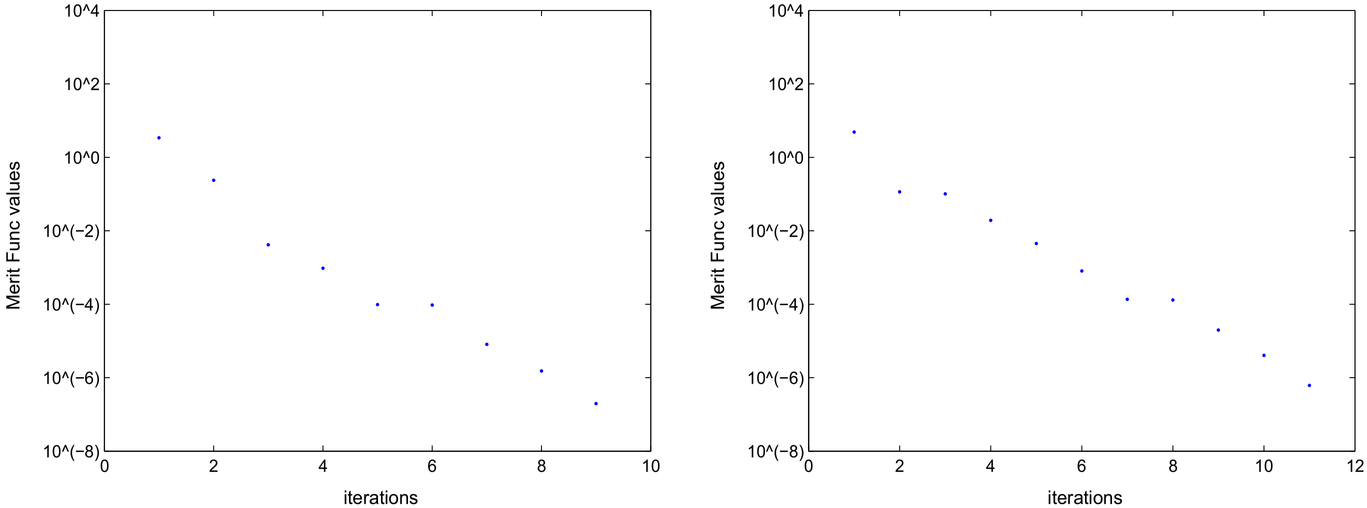

Fig. 2 Convergence behavior of linear case (*left*) and nonlinear case (*right*) of test problem 1 based on ϕ_1 with $p = 3, \alpha = 1$ and starting point (c)

Proof

Journal: **40314** Article No.: **0328** ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:**2016/2/26** Pages: **26** Layout: **Small**

Author Proof

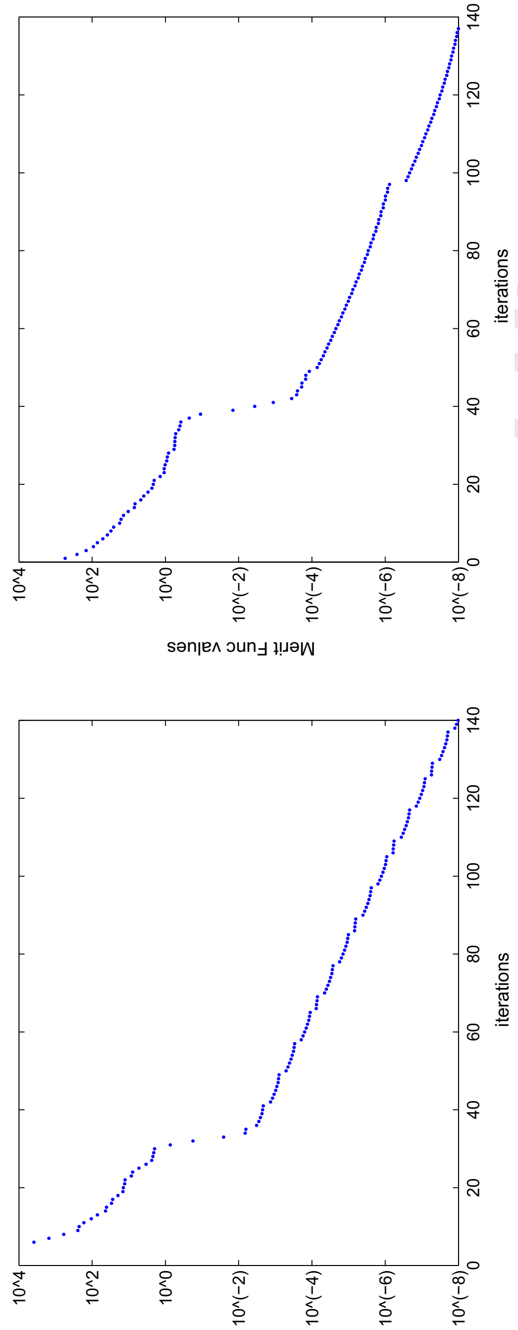

Fig. 3 Convergence behavior of Example 2 based on ϕ_p with $p = 4$ and starting point (a) (*left*) and test problem 2 based on $\phi_{\theta p}$ with $p = 1.5, \theta = 0.75$ starting point (b) (*right*)

proof

Journal: **40314** Article No.: **0328** TYPESET DISK LE CP Disp.:**2016/2/26** Pages: **26** Layout: **Small**

Final remarks

In this paper, starting with C^1 functions, we give the sufficient conditions on the functions f and g so that we can guarantee that stationary points of the merit function solve the generalized complementarity problem GCP(f, g).

For continuously differentiable functions, the nonsingularity of $\{\nabla g\}$ is very important in an algorithmic point of view and studying the error bounds for GCP(f, g). Thus, the nonsingularity of $\{\nabla g\}$ is not restrictive.

We consider a generalized complementarity problem based on generalized Fischer-Burmeister function and its generalizations corresponding to C^1 functions, with an associated GCP function Φ and a merit function $\Psi_*(x) = \frac{1}{2}||\Phi_*||^2$. We show under certain regularity conditions the global/local minimum or a stationary point of Ψ_* is a solution of GCP(f, g). Our results give various results for generalized complementarity problem when p-norm replaces by 2-norm (or when p is an integer greater than 2). Also, when $g(x) = x$, our results further give a unified/generalization treatment of such results for the nonlinear complementarity problem based on generalized Fischer-Burmeister function and its generalizations.

Moreover, we present a descent algorithm for GCP(f, g) and show a result on the global convergence of a descent algorithm for solving generalized complementarity problem. Furthermore, we present some preliminary numerical results. The numerical results suggest that different combinations of GCP functions and parameters p, α, and θ can yield improved performance in the descent direction algorithm presented in this paper. The two measurements of the effectiveness of the algorithm include the number of iterations taken to find a solution and the final value of the merit function at the end of the algorithm. The numerical results suggested that having a lower p parameter may improve convergence behavior and that GCP functions 3 and 4 perform best across our test problem set. To the best of our knowledge, solving GCP(f, g) on the basis of generalized Fischer-Burmeister function and its generalizations seems to be new.

It should be pointed out that our implementation is still in an early stage. The following directions in the future research can be pursued to improve the current implementation:

- Apply a quasi-Newton method for GCP functions based on generalized Fischer function.
- Apply a conjugate gradient method with descent direction to GCP based on generalized Fischer function.
- Can we establish the convergence of quasi-Newton method and conjugate gradient method?
- Implement a descent method, conjugate gradient and quasi-Newton method to more examples for GCP from Andreani et al. (2002), Jiang et al. (1998).

Acknowledgments The authors are much indebted to the anonymous referee for the constructive comments and useful suggestions which improved the presentation of the paper considerably. The research of the 1st author is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC). The postdoctoral fellowship of the 2nd author is supported by NSERC.

References

Aashtiani H, Magnanti T (1981) Equilibrium on a congested transportation network. SIAM J Algebr Discrete Methods 42:213–226

Agdeppa RP, Yamashita N, Fukushima M (2007) The traffic equilibrium problem with nonadditive costs and its monotone mixed complementarity problem formulation. Transp Res Part B 41:862874

Andreani R, Friedlander A, Santos SA (2002) On the resolution of the generalized nonlinear complementarity problem. SIAM J Optim 12(2):303–321

Author Proof

uncorrected proof

Author Proof

Chen J-S, Huang Z-H, She C-Y (2011) A new class of penalized NCP-functions and its properties. Comput Optim Appl 50:49–73

Chen M, Bernstein DH, Chien SIJ, Mouskos K (1999) Simplified formulation of the toll design problem. Transp Res Rec 1667:8895

Chen J-S (2006) The semismooth-related properties of a merit function and a descent method for the nonlinear complementarity problem. J Glob Optim 36:565580

Chen Jein-shan (2007) On some NCP-functions based on the generalized Fischer-Burmeister function. Asia Pac J Oper Res 24(3):401–420

Chen J-S, Gao H-T, Pan S-H (2009) An R-linearly convergent derivative-free algorithm for nonlinear complementarity problems based on the generalized Fischer-Burmeister merit function. J Comput Appl Math 232:455–471

Chen J-S, Pan S-H (2008) A family of NCP functions and a descent method for the nonlinear complementarity problems. Comput Optim Appl 40:389–404

Company R, Egorova VN, Jdar L (2014) Solving American option pricing models by the front fixing method: numerical analysis and computing. Abstr Appl Anal, Article ID 146745

Cottle RW, Giannessi F, Lions J-L (eds) (1980) Variational inequalities and complementarity problems: theory and applications. Wiley, New York

Cottle RW, Pang J-S, Stone RE (1992) The linear complementarity problem. Academic Press, Boston

Dirkse SP, Ferris M (1994) MCPLIB: a collection of nonlinear mixed complementarity problems. Optim Methods Softw 5:319–345

Di Pillo G, Giannessi F (eds) (1996) Nonlinear optimization and applications. Plenum Press, New York

Facchinei F, Kanzow C (1997) On unconstrained and constrained stationary points of the implicit Lagrangian. J Optim Theory Appl 92:99–115

Facchinei F, Pang J-S (2003) Finite dimensional variational inequalities and complementarity problems. Springer-Verlag, New York

Facchinei F, Soares J (1997) A new merit function for nonlinear complementarity problems and related algorithm. SIAM J Optim 7:225–247

Feng L, Linetsky V, Morales JL, Nocedal J (2011) On the solution of complementarity problems arising in American options pricing. Optim Methods Softw 26(4–5):813–825

Ferris MC, Pang J-S (1997b) Engineering and economic applications of complementarity problems. SIAM Rev 39:669–713

Ferris MC, Pang J-S (eds) (1997a) Complementarity and variational problems: state of the art. SIAM, Philadelphia

Ferris MC, Ralph D (1995) Projected gradient methods for nonlinear complementarity problems via normal maps. In: Du DZ, Qi L, Womersley RS (eds) Recent advances in nonsmooth optimization. World Scientific Publishers, Singapore, pp 57–87

Fischer A (1997) Solution of monotone complementarity problems with locally Lipschitzian functions. Math Program 76:513–532

Fischer A (1998) A New constrained optimization reformulation for complementarity problems. J Optim Theory Appl 97:105–117

Gabriel SA, Bernstein D (1997) The traffic equilibrium problem with nonadditive path costs. Transp Sci 31:337–348

Galántai A (2012) Properties and construction of NCP functions. Comput Optim Appl 52:805–824

Geiger C, Kanzow C (1996) On the resolution of monotone complementarity problems. Comput Optim 5:155–173

Gu W-Z, Tawhid MA (2014) Generalized complementarity problems based on generalized Fischer-Burmeister functions as unconstrained optimization. Adv Model Optim 16(2):269–284

Harker PT, Pang J-S (1990) Finite dimension variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. Math Program 48:161–220

Hu S-L, Huang Z-H, Chen J-S (2009) Properties of a family of generalized NCP-functions and a derivative free algorithm for complementarity problems. J Comput Appl Math 230:68–82

Huang J, Pang J-S (1998) Option pricing and linear complementarity. J Comput Finance 2:31–60

Hyer DH, Isac G, Rassias TM (1997) Topics in nonlinear analysis and applications. World scientific Publishing Company, Singapore

Isac G (1992) Complementarity problems, Lecture Notes in Mathematics, vol 1528. Springer Verlag, Berlin

Jiang H (1996) Unconstrained minimization approaches to nonlinear complementarity problems. J Glob Optim 9:169–181

Jiang H, Fukushima M, Qi L, Sun D (1998) A trust region method for solving generalized complementarity problems. SIAM J Optim 8(1):140–157

uncorrected proof

Kanzow C (1996) Nonlinear complementarity as unconstrained optimization. J Optim Theory Appl 88:139–155
Lo HK, Chen A (2000) Traffic equilibrium problem with route-specific costs: formulation and algorithms. Transp Res Part B 34:493–513
Luca TD, Facchinei F, Kanzow C (1996) A semismooth equation approach to the solution of nonlinear complementarity problems. Math Program 75:407–439
Mangasarian OL, Solodov MV (1993) Nonlinear complementarity as unconstrained and constrained minimization. Math Program 62:277–297
McKean HP (1965) Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics. Ind Manag Rev 6:3239
Noor MA (1993) General nonlinear complementarity problems. In: Srivastava HM, Rassias TM (eds) Analysis, geometry, and groups: a riemann legacy volume. Hadronic Press, Palm Harbor, pp 337–371
Noor MA (1988) Quasi-complementarity problem. J Math Anal Appl 130:344–353
Ortega JM, Rheinboldt WC (1970) Iterative solution of nonlinear equations in several variables. Academic Press, New York, San Francisco, London
Outrata JV, Zowe J (1995) A Newton method for a class of quasi-variational inequalities. Comput Optim Appl 4:5–21
Pang JS (1981) The implicit complementarity problem. In: Mangasarian OL, Meyer RR, Robinson SM (eds) Nonlinear programming. Academic Press, New York, pp 487–518
Patriksson M (1994) The traffic assignment problem: models and methods. VSP, Utrecht
Patriksson M (2004) Algorithms for computing traffic equilibria. Netw Spatial Econ 4:2338
Sheffi Y (1985) Urban transportation networks: equilibrium analysis with mathematical programming methods. Prentice Hall, New Jersey
Tawhid MA (2008) An application of H-differentiability to nonnegative and unrestricted generalized complementarity. Comput Optim Appl 39:51–74
Tseng P (1996) Growth behavior of a class of merit functions for the nonlinear complementarity problem. J Optim Theory Appl 89:17–37
van Moerbeke P (1976) On optimal stopping and free boundary problems. Arch Ration Mech Anal 60(2):101148
Wilmott P, Howison S, Dewynne J (1995) The mathematics of financial derivatives. Cambridge University Press, Cambridge
Xu M, Gao ZY (2011) A complementary formulation for traffic equilibrium problem with a new nonadditive route cost. Sci China Technol Sci 54(9):2525–2530
Yamada K, Yamashita N, Fukushima M (2000) A new derivative-free descent method for the nonlinear complementarity problems. In: Pillo GD, Giannessi F (eds) Nonlinear optimization and related topics. Kluwer Academic Publishers, Netherlands, pp 463–487
Yamashita N, Fukushima M (1995) On stationary points of the implicit Lagrangian for nonlinear complementarity problems. J Optim Theory Appl 84:653–663

Author Proof
uncorrected proof

Author Proof

Journal: 40314
Article: 328

Springer
the language of science

Author Query Form

Please ensure you fill out your response to the queries raised below and return this form along with your corrections

Dear Author

During the process of typesetting your article, the following queries have arisen. Please check your typeset proof carefully against the queries listed below and mark the necessary changes either directly on the proof/online grid or in the 'Author's response' area provided below

Query	Details required	Author's response
1.	Kindly check and confirm the edit in the sentence "We give some examples ..." in the manuscript.	
2.	Kindly check and confirm the edit in the term "Lipschitz continuous" in the manuscript.	
3.	Kindly check the clarity of the sentence "To test the effectiveness of ..." in the manuscript.	
4.	This sentence has been slightly modified. Please check that the meaning is still correct, and amend if necessary. "Two starting points are used:"	
5.	Please provide a definition for the significance of [bold values] in the table 3.	
6.	Please check and confirm if the inserted citation of Tables 6 and 9 are correct. If not, please suggest an alternate citation. Please note that tables should be cited sequentially in the text.	
7.	This sentence has been slightly modified. Please check that the meaning is still correct, and amend if necessary. "The numerical results suggested ..."	

uncorrected proof