ORTHOGONAL POLYNOMIALS AND CONNECTION TO GENERALIZED MOTZKIN NUMBERS FOR HIGHER-ORDER EULER POLYNOMIALS

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Abstract. We study the higher-order Euler polynomials and give the corresponding monic orthogonal polynomials, which are Meixner-Pollaczek polynomials with certain arguments and constant factors. Moreover, we obtain a connection to the generalized Motzkin number, which leads to a new recurrence formula and a matrix representation for the higher-order Euler polynomials.

1. Introduction

The *Bernoulli numbers* B_n and *Euler numbers* E_n, defined by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}\quad\text{and}\quad\frac{2}{e^z+e^{-z}}=\sum_{n=0}^{\infty}E_n\frac{z^n}{n!},\tag{1.1}$$

belong to the most important numbers, with various applications in number theory and also other fields of mathematics. These sequences of numbers satisfy numerous properties, including connections with certain matrices and determinants. See, e.g, [4]. A new such connection is as follows. Define the doubly infinite band matrix

$$\mathbf{E}:=\begin{pmatrix}0 & -1 & 0 & 0 & \cdots & 0 & \cdots\\ 1 & 0 & -4 & 0 & \cdots & 0 & \cdots\\ 0 & 1 & 0 & \ddots & \ddots & \vdots & \cdots\\ 0 & 0 & 1 & \ddots & -m^2 & 0 & \cdots\\ \vdots & \vdots & \vdots & \ddots & 0 & -(m+1)^2 & \cdots\\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots\\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots\end{pmatrix}.$$

If for each $n\geq 1$, we take the n-th power of the upper left $n\times n$ submatrix of $\mathbf{E}$, then the upper left entry of this power will be E_n. In this way, we can easily obtain $E_1=0$, $E_2=-1$, $E_3=0$, and $E_4=5$, which are consistent with the values that can be obtained from the generating function (1.1).

It is one of the objectives of this paper to explain this phenomenon. In fact, we will generalize it in two directions: one for *Euler polynomials of higher order*, which contains the ordinary *Euler polynomials* $E_n(x)$ and Euler numbers E_n as special cases; and the other for *Bernoulli polynomials* $B_n(x)$ and Bernoulli numbers B_n.

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In this process, we encounter the generalized Motzkin numbers, special orthogonal polynomials, and some probabilistic methods that can be applied in number theory.

The *Euler polynomials of order* p, denoted by $E_n^{(p)}(x)$, are defined by

$$\left(\frac{2}{e^z+1}\right)^p e^{xz} = \sum_{n=0}^{\infty} E_n^{(p)}(x)\frac{z^n}{n!}, \tag{1.2}$$

with special values $E_n^{(1)}(x) = E_n(x)$ and $E_n^{(1)}(1/2) = E_n(1/2) = E_n/2^n$. (See, e.g., [13, Entries 24.16.3 and 24.2.9].) The first several terms of $E_n^{(p)}(x)$ are as follows.

	$p=1$	$p=2$	$p=3$
$n=0$	1	1	1
$n=1$	$x-\frac{1}{2}$	$x-1$	$x-\frac{3}{2}$
$n=2$	x^2-x	$x^2-2x+\frac{1}{2}$	$x^2-3x+\frac{3}{2}$
$n=3$	$x^3-\frac{3}{2}x^2+\frac{1}{4}$	$x^3-3x^2+\frac{3}{2}x+\frac{1}{2}$	$x^3-\frac{9}{2}x^2+\frac{9}{2}x$
$n=4$	x^4-2x^3+x	$x^4-4x^3+3x^2+2x-1$	$x^4-6x^3+9x^2-3$

Table 1. $E_n^{(p)}(x)$ for $0 \le n \le 4$ and $1 \le p \le 3$

Our first result here is to give the monic orthogonal polynomials with respect to $E_n^{(p)}(x)$. Given a sequence m_n, we study the monic orthogonal polynomials $P_n(y)$ with respect to m_n. The orthogonality means, for integers r and n with $0 \le r < n$,

$$y^r P_n(y)|_{y^k=m_k} = 0, \tag{1.3}$$

where the left-hand side means expanding the polynomial and evaluating as $y^k = m_k$ on each power of y. In addition, P_n satisfies a three-term recurrence [14, p. 47]: for certain sequences s_n and t_n, $P_0(y) = 1$, $P_1(y) = y - s_0$, and when $n \ge 1$,

$$P_{n+1}(y) = (y - s_n)P_n(y) - t_n P_{n-1}(y). \tag{1.4}$$

After Touchard [16, eq. 44] computed the monic orthogonal polynomials with respect to the B_n, Carlitz [3, eq. 4.7] and also with Al-Salam [1, p. 93] gave the monic orthogonal polynomials, denoted by $Q_n(y)$, with respect to E_n. More precisely, they obtained $Q_0(y) = 1$, $Q_1(y) = y$ and for $n \ge 1$,

$$Q_{n+1}(y) = yQ_n(y) + n^2 Q_{n-1}(y). \tag{1.5}$$

Now, let $\Omega_n^{(p)}(y)$ be the monic orthogonal polynomials with respect to $E_n^{(p)}(x)$, i.e., similarly as (1.3), for integers r and n, with $0 \le r < n$,

$$y^r \Omega_n^{(p)}(y)|_{y^k=E_k^{(p)}(x)} = 0. \tag{1.6}$$

Our first result is to give the recurrence of $\Omega_n^{(p)}(y)$.

Theorem 1. *For integer* $p \ge 1$*, we have* $\Omega_0^{(p)}(y) = 1$, $\Omega_1^{(p)}(y) = y - x + p/2$ *and*

$$\Omega_{n+1}^{(p)}(y) = \left(y - x + \frac{p}{2}\right)\Omega_n^{(p)}(y) + \frac{n(n+p-1)}{4}\Omega_{n-1}^{(p)}(y). \tag{1.7}$$

See Example 10 to illustrate the orthogonality (1.6) for $p = n = 2$. Furthermore, Theorem 1 links $E_n^{(p)}(x)$ to the generalized Motzkin numbers, defined next.

Definition 2. Given arbitrary sequences σ_k and τ_k, the *generalized Motzkin numbers* $M_{n,k}$ are defined by $M_{0,0} = 1$ and for $n > 0$ by the recurrence

$$M_{n+1,k} = M_{n,k-1} + \sigma_k M_{n,k} + \tau_{k+1} M_{n,k+1}, \tag{1.8}$$

where $M_{n,k} = 0$ if $k > n$ or $k < 0$. (See also [12, eq. 3].)

The second result identifies $E_n^{(p)}(x)$ as the generalized Motzkin numbers, which allows us to endow new recurrence, as (1.8), and matrix representation, for $E_n^{(p)}(x)$.

Theorem 3. *Let $\mathfrak{E}_{n,k}^{(p)}$ be the generalized Motzkin numbers with special choices $\sigma_k = x - p/2$ and $\tau_k = -k\,(k+p-1)\,/4$. Then, $\mathfrak{E}_{n,0}^{(p)} = E_n^{(p)}(x)$.*

To prove both Theorem 1 and Theorem 3, we shall organize this paper as follows.

In Section 2, we first review some basic definitions and properties on random variables, orthogonal polynomials, and also the probabilistic interpretation for $E_n^{(p)}(x)$, viewing them as moments of a certain random variable. Next, instead of proving the recurrence (1.7), we identify $\Omega_n^{(p)}(y)$ as the Meixner-Pollaczek polynomials, whose definition and important properties will also be introduced in this section.

In Section 3, after introducing a combinatorial interpretation of the generalized Motzkin numbers, we present two continued fractions expressions. This leads to a general theorem, Theorem 13, identifying moments of a random variable and the generalized Motzkin numbers. Then, we see that Theorem 3 is just a special case on $E_n^{(p)}(x)$. Moreover, the combinatorial interpretation as weighted lattice paths provides a matrix representation for $E_n^{(p)}(x)$. In the end, an example presents a connection between the Euler numbers and Catalan numbers.

In the last section, Section 4, we give analogues for Bernoulli polynomials and a conjecture for the higher-order Bernoulli polynomials.

2. Orthogonal polynomials for higher-order Euler polynomials

2.1. Preliminaries.

We first recall some necessary definitions and classical results on random variables.

Given an arbitrary random variable X on $\mathbb{R}$, with probability density function $p(t)$ and moments m_n, namely, $m_n = \mathbb{E}[X^n] = \int_{\mathbb{R}} t^n p(t)\mathrm{d}t$, one can consider the *monic orthogonal polynomials* with respect to X, denoted by $P_n(y)$, which are monic polynomials with degree $\deg P_n = n$. For positive integers u and v,

$$\mathbb{E}\left[P_u(X)P_v(X)\right] = \int_{\mathbb{R}} P_u(t)P_v(t)p(t)\mathrm{d}t = c_u\delta_{u,v}$$

(see [14, eq. 2.20]), where c_u are constants depending on u and $\delta_{u,v}$ is the Kronecker delta function, which gives 1 if $u = v$, and 0 if $u \neq v$. Equivalently, the orthogonality can be expressed as a system of equations: for integers r and n with $0 \leq r < n$,

$$\mathbb{E}\left[X^r P_n(X)\right] = \int_{\mathbb{R}} t^r P_n(t)\mathrm{d}t = y^r P_n(y)|_{y^k = m_k} = 0,$$

which is the same as (1.3). The three-term recurrence of $P_n(y)$ is stated in (1.4), with $P_0(y) = 1$ and $P_1(y) = y - s_0$.

If X' is another random variable independent of X with moments m'_n, then for the random variable $X + X'$ we have

$$\mathbb{E}\left[(X+X')^n\right] = \sum_{k=0}^{n}\binom{n}{k} m_k m'_{n-k} = (y_1+y_2)^n\,|_{y_1^k = m_k, y_2^k = m'_k}. \tag{2.1}$$

The next lemma gives the moments and the monic orthogonal polynomials after shifting or scaling X, which is crucial in the proof of Theorem 1.

Lemma 4. *Let C and c be constants.*

1. *For the shifted random variable $X+c$, the corresponding moments are*

$$\mathbb{E}\left[(X+c)^n\right] = \sum_{k=0}^{n} \binom{n}{k} m_k c^{n-k}$$

and the monic orthogonal polynomials, denoted by $\bar{P}_n(y)$, satisfy $\bar{P}_0(y)=1$, $\bar{P}_1(y) = y - s_0 - c$ and for $n \geq 1$,

$$\bar{P}_{n+1}(y) = (y - s_n - c)\bar{P}_n(y) - t_n \bar{P}_{n-1}(y). \tag{2.2}$$

2. *For the scaled random variable CX, the moments are $\mathbb{E}\left[(CX)^n\right] = C^n m_n$, and the monic orthogonal polynomials, denoted by $\tilde{P}_n(y)$, satisfy $\tilde{P}_0(y) = 1$, $\tilde{P}_1(y) = y - Cs_0$ and for $n \geq 1$,*

$$\tilde{P}_{n+1}(y) = (y - Cs_n)\tilde{P}_n(y) - C^2 t_n \tilde{P}_{n-1}(y). \tag{2.3}$$

Proof. Computations for the moments are straightforward. We only consider the monic orthogonal polynomials.

1. For $X+c$, notice that $\bar{P}_n(y) := P_n(y-c)$ satisfies

$$\mathbb{E}\left[\bar{P}_u(X+c)\bar{P}_v(X+c)\right] = \mathbb{E}\left[\bar{P}_u(X)\bar{P}_v(X)\right] = c_u \delta_{u,v}.$$

The recurrence (2.2) follows by shifting $y \mapsto y - c$ in (1.4).

2. Similarly, for CX, $\tilde{P}_n(y) := C^n P_n(y/C)$. □

Next, we recall the probabilistic interpretation for $E_n^{(p)}(x)$. See, e.g., [9, eq. 2.6].

Let L_E be a random variable with density function $p_E(t) := \operatorname{sech}(\pi t)$ on $\mathbb{R}$. Also consider a sequence of independent and identically distributed (i. i. d.) random variables $(L_{E_i})_{i=1}^{p}$ with each L_{E_i} having the same distribution as L_E. Then $E_n^{(p)}(x)$ is the n-th moment of a certain random variable:

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + \sum_{i=1}^{p} iL_{E_i} - \frac{p}{2}\right)^n\right].$$

For simplicity, we denote $\epsilon_i = iL_{E_i}$ and $\epsilon^{(p)} := \sum_{i=1}^{p} \epsilon_i$. Then $(\epsilon_i)_{i=1}^{p}$ is also an i. i. d. sequence. Moreover,

$$E_n^{(p)}(x) = \mathbb{E}\left[\left(x + \epsilon^{(p)} - \frac{p}{2}\right)^n\right]. \tag{2.4}$$

The *higher-order Euler numbers* are usually defined as $E_n^{(p)} := E_n^{(p)}(0)$, for $p > 1$. We next define another sequence of numbers, related to $E_n^{(p)}(x)$ and $E_n^{(p)}$, as follows.

Definition 5. Define the sequence $\bar{E}_n^{(p)}$ by the exponential generating function

$$\left(\frac{2}{e^z + e^{-z}}\right)^p = \sum_{n=0}^{\infty} \bar{E}_n^{(p)} \frac{z^n}{n!}.$$

From (1.2) and (2.4), we see that

$$\bar{E}_n^{(p)} = 2^n E_n^{(p)}\left(\frac{p}{2}\right), \quad \bar{E}_n^{(1)} = E_n, \quad \text{and} \quad \bar{E}_n^{(p)} = \mathbb{E}\left[\left(2\epsilon^{(p)}\right)^n\right]. \tag{2.5}$$

As stated in Section 1, the orthogonal polynomials with respect to E_n, denoted by $Q_n(y)$ satisfy the recurrence (1.5). In fact, the next definition shows that $Q_n(y)$ are basically the Meixner-Pollaczek polynomials; see, e.g., [10, eq. 9.7.1].

Definition 6. The *Meixner-Pollaczek polynomials* are defined by

$$P_n^{(\lambda)}(y;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(\begin{matrix} -n, \lambda+iy \\ 2\lambda \end{matrix} \middle| 1-e^{-2i\phi}\right),$$

where $(x)_n := x(x+1)(x+2)\cdots(x+n-1)$ is the Pochhammer symbol and ${}_2F_1$ is the hypergeometric function [13, Entries 15.1.1 and 15.2.1].

Following a similar computation as that in, e.g., [7, p. 1], we see

$$Q_n(y) := i^n n! P_n^{(\frac{1}{2})}\left(\frac{-iy}{2}; \frac{\pi}{2}\right). \tag{2.6}$$

Two important properties of $P_n^{(\lambda)}(y;\phi)$ are listed in the following proposition.

Proposition 7. *The Meixner-Pollaczek polynomials* $P_n^{(\lambda)}(y;\phi)$ *satisfy the recurrence*

$$(n+1)P_{n+1}^{(\lambda)}(y;\phi) = 2(y\sin\phi+(n+\lambda)\cos\phi)P_n^{(\lambda)}(y;\phi)-(n+2\lambda-1)P_{n-1}^{(\lambda)}(y;\phi); \tag{2.7}$$

and the convolution formula

$$P_n^{(\lambda+\mu)}(y_1+y_2,\phi) = \sum_{k=0}^{n} P_k^{(\lambda)}(y_1,\phi)\, P_{n-k}^{(\mu)}(y_2,\phi). \tag{2.8}$$

For proofs and further details of the proposition above, see [10, eq. 9.7.3] and [2, p. 17], respectively.

2.2. **Proof of Theorem 1.** The following theorem gives an explicit expression of $\Omega_n^{(p)}(y)$, which implies Theorem 1, by (2.7).

Theorem 8. *For positive integers n and p, we have*

$$\Omega_n^{(p)}(y) = \frac{i^n n!}{2^n} P_n^{(\frac{p}{2})}\left(-i\left(y-x+\frac{p}{2}\right); \frac{\pi}{2}\right).$$

To prove this result, we need the following lemma.

Lemma 9. *Let $Q_n^{(p)}(y)$ be the monic orthogonal polynomials with respect to $\bar{E}_n^{(p)}$. Then,*

$$Q_n^{(p)}(y) = i^n n! P_n^{(\frac{p}{2})}\left(-\frac{iy}{2}; \frac{\pi}{2}\right). \tag{2.9}$$

Proof. We shall prove this by induction on the order p. Obviously, (2.9) coincides with (2.6) when $p=1$. If $p>1$, we write $p=p_1+p_2$, with both $p_1,\ p_2 \geq 1$. By inductive hypothesis, $Q_n^{(p_i)}(y)$, defined by (2.9), are the monic orthogonal polynomials with respect to $\bar{E}_n^{(p_i)}$, for $i=1,\ 2$. Now, we write $y=y_1+y_2$, so that by (2.8),

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}Q_k^{(p_1)}(y_1)Q_{n-k}^{(p_2)}(y_2)\\
=&\sum_{k=0}^{n}\binom{n}{k}\left[i^k k! P_k^{\left(\frac{p_1}{2}\right)}\left(-\frac{iy_1}{2};\frac{\pi}{2}\right)\right]\left[i^{n-k}(n-k)!P_{n-k}^{\left(\frac{p_2}{2}\right)}\left(-\frac{iy_2}{2};\frac{\pi}{2}\right)\right]\\
=&i^n n!\sum_{k=0}^{n}P_k^{\left(\frac{p_1}{2}\right)}\left(-\frac{iy_1}{2};\frac{\pi}{2}\right)P_{n-k}^{\left(\frac{p_2}{2}\right)}\left(-\frac{iy_2}{2};\frac{\pi}{2}\right)\\
=&i^n n! P_n^{\left(\frac{p}{2}\right)}\left(-\frac{iy}{2};\frac{\pi}{2}\right)=Q_n^{(p)}(y).
\end{aligned}$$

To show the orthogonality, we consider integers r and n with $0\le r<n$. Then

$$\begin{aligned}
y^r Q_n^{(p)}(y) &= (y_1+y_2)^r\sum_{k=0}^{n}\binom{n}{k}Q_k^{(p_1)}(y_1)Q_{n-k}^{(p_2)}(y_2)\\
&=\left(\sum_{l=0}^{n}\binom{n}{l}y_1^l y_2^{r-l}\right)\left(\sum_{k=0}^{n}\binom{n}{k}Q_k^{(p_1)}(y_1)Q_{n-k}^{(p_2)}(y_2)\right)\\
&=\sum_{l=0}^{n}\sum_{k=0}^{n}\binom{n}{l}\binom{n}{k}\left(y_1^l Q_k^{(p_1)}(y_1)\right)\left(y_2^{r-l}Q_{n-k}^{(p_2)}(y_2)\right).
\end{aligned}$$

From (2.1) and the fact that $\bar{E}_n^{(p)}=\mathbb{E}\left[\left(2\epsilon^{(p)}\right)^n\right]=\mathbb{E}\left[\left(2\epsilon^{(p_1)}+2\epsilon^{(p_2)}\right)^n\right]$, we have

$$\begin{aligned}
&y^r Q_n^{(p)}(y)\big|_{y^s=\bar{E}_s^{(p)}}\\
=&(y_1+y_2)^r Q_n^{(p)}(y_1+y_2)\big|_{y_1^s=\bar{E}_s^{(p_1)},y_2^s=\bar{E}_s^{(p_2)}}\\
=&\sum_{l=0}^{n}\sum_{k=0}^{n}\binom{n}{l}\binom{n}{k}\left(y_1^l Q_k^{(p_1)}(y_1)\right)\Big|_{y_1^s=\bar{E}_s^{(p_1)}}\left(y_2^{r-l}Q_{n-k}^{(p_2)}(y_2)\right)\Big|_{y_2^s=\bar{E}_s^{(p_2)}}.
\end{aligned}$$

Since $l+(r-l)=r<n=k+(n-k)$ for each term in the sum above, either $l<k$ or $r-l<n-k$ holds, implying the orthogonality: $y^r Q_n^{(p)}(y)\big|_{y^s=\bar{E}_s^{(p)}}=0$. □

Proof of Theorem 8. From (2.4) and (2.5), we see

$$E_n^{(p)}(x)=\mathbb{E}\left[\left(x+\epsilon^{(p)}-\frac{p}{2}\right)^n\right]=\mathbb{E}\left[\left(x-\frac{p}{2}+\frac{1}{2}\cdot\left(2\epsilon^{(p)}\right)\right)^n\right],$$

where, as shown above, $\bar{E}_n^{(p)}=\mathbb{E}\left[\left(2\epsilon^{(p)}\right)^n\right]$. Then, we apply Lemma 4 twice, for $C=1/2$ and $c=x-p/2$, to obtain

$$\Omega_n^{(p)}(y)=\frac{1}{2^n}Q_n^{(p)}\left(2\left(y-x+\frac{p}{2}\right)\right)=\frac{i^n n!}{2^n}P_n^{\left(\frac{p}{2}\right)}\left(-i\left(y-x+\frac{p}{2}\right);\frac{\pi}{2}\right),$$

which completes the proof. □

To conclude this section, we present an example to illustrate the orthogonality relation (1.6).

Example 10. When $p=2$, we see by (1.7)

$$\Omega_2^{(2)}(y)=\left(y-x+\frac{2}{2}\right)^2+\frac{1(1+2-1)}{4}=y^2-2\left(x-1\right)y+\left(x-1\right)^2+\frac{1}{2}.$$

Using Table 1, we have

$$\begin{aligned} y^0\Omega_2^{(2)}(y)|_{y^k=E_k^{(2)}(x)} &= y^2-2\,(x-1)\,y+(x-1)^2+\frac{1}{2}|_{y^k=E_k^{(2)}(x)} \\ &= x^2-2x+\frac{1}{2}-2\,(x-1)^2+(x-1)^2+\frac{1}{2}=0, \end{aligned}$$

and similarly

$$\begin{aligned} &y\Omega_2^{(2)}(y)|_{y^k=E_k^{(2)}(x)} \\ =&y^3-2\,(x-1)\,y^2+(x-1)^2\,y+\frac{y}{2}|_{y^k=E_k^{(2)}(x)} \\ =&x^3-3x^2+\frac{3}{2}x+\frac{1}{2}-2\,(x-1)\left(x^2-2x+\frac{1}{2}\right)+(x-1)^3+\frac{x-1}{2}=0. \end{aligned}$$

This confirms (1.6) for $p=n=2$.

3. Connection to generalized Motzkin numbers

3.1. **Preliminaries.** Recall the definition of the generalized Motzkin numbers in Definition 2. In fact, $M_{n,k}$ counts the number of certain weighted lattice paths, called *Motzkin paths* [6, p. 319]. More specifically, consider the paths with the following restrictions:

1. all paths lie within the first quadrant;
2. only allow three types of paths:
   a) horizontal path α_k from (j,k) to $(j+1,k)$;
   b) diagonally up path β_k from (j,k) to $(j+1,k+1)$;
   c) and diagonally down path γ_k from (j,k) to $(j+1,k-1)$;
3. associate each type of the paths to different weights as $\alpha_k\mapsto 1$, $\beta_k\mapsto\sigma_k$, and $\gamma_k\mapsto\tau_k$, which we shall denote by the weight triple $(1,\sigma_k,\tau_k)$.

Then, $M_{n,k}$ counts the number of $(1,\sigma_k,\tau_k)$-weighted paths from $(0,0)$ to (n,k).

The next result shows that when $k=0$, the generating function of $M_{n,0}$ admits a form of continued fractions (called *J(acobi)-fractions*). See, e.g., [6, p. 324].

Theorem 11. *For the generalized Motzkin numbers $M_{n,k}$ defined by (1.8), we have*

$$\sum_{n=0}^{\infty} M_{n,0}z^n=\cfrac{1}{1-\sigma_0 z-\cfrac{\tau_1 z^2}{1-\sigma_1 z-\cfrac{\tau_2 z^2}{1-\sigma_2 z-\cdots}}}. \tag{3.1}$$

A similar expression is known for moments. See, e.g., [11, pp. 20–21].

Theorem 12. *Let X be an arbitrary random variable, with moments m_n and monic orthogonal polynomials $P_n(y)$ satisfying the recurrence (1.4), involving two sequences s_n and t_n. Then, we have*

$$\sum_{n=0}^{\infty} m_n z^n=\cfrac{m_0}{1-s_0 z-\cfrac{t_1 z^2}{1-s_1 z-\cfrac{t_2 z^2}{1-s_2 z-\cdots}}}. \tag{3.2}$$

Combining (3.2) and (3.1) leads to the following general theorem.

Theorem 13. *Let the two sequences s_n and t_n be the ones appearing in the recurrence (1.4), for some random variable X, and assume that $m_0=1$. Also define the generalized Motzkin number sequence $M_{n,k}$ by letting $\sigma_k=s_k$ and $\tau_k=t_k$ in (1.8). Then, $M_{n,0}$ gives the moments of X, i.e., $M_{n,0}=m_n=\mathbb{E}[X^n]$.*

Remark. The condition $m_0 = 1$ in Theorem 13 is usually guaranteed by normalization of the density function.

3.2. **Proof and Applications of Theorem 3.**

Proof of Theorem 3. We apply Theorem 13 to the random variable $x + \epsilon^{(p)} - p/2$, whose moments are $E_n^{(p)}(x)$, we directly prove Theorem 3. □

The combinatorial interpretation for $E_n^{(p)}(x)$ can now be used to to obtain the following matrix representation.

Theorem 14. *Define the infinite dimensional matrix*

$$RE^{(p)} := \begin{pmatrix} x-\frac{p}{2} & -\frac{p}{4} & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{p}{2} & -\frac{p+1}{2} & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{p}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\frac{n(n+p-1)}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{p}{2} & -\frac{(n+1)(n+p)}{4} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix},$$

and let $RE_m^{(p)}$ be the left upper $m \times m$ block of $RE^{(p)}$. Then for any nonnegative integer $n \le m$, the left upper entry of $\left(RE_m^{(p)}\right)^n$ gives $E_n^{(p)}(x)$, i.e.,

$$\left[\left(RE_m^{(p)}\right)^n\right]_{1,1} = E_n^{(p)}(x).$$

Example 15. The case for $p = 1$ and $x = 1/2$ is shown in the Introduction. Now, let $p = 2$ and $m = 4$. Then

$$RE_4^{(2)} := \begin{pmatrix} x-1 & -1/2 & 0 & 0 \\ 1 & x-1 & -3/2 & 0 \\ 0 & 1 & x-1 & -3 \\ 0 & 0 & 1 & x-1 \end{pmatrix}$$

and

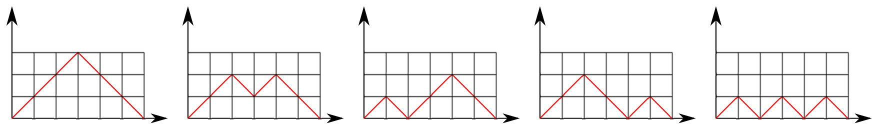
$$\left(RE_4^{(2)}\right)^3 = \begin{pmatrix} x^3-3x^2+\frac{3}{2}x+\frac{1}{2} & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix},$$

confirming $E_3^{(2)}(x) = x^3 - 3x^2 + 3x/2 + 1/2$. (See Table 1.)

Example 16. Recall $E_n = 2^n E_n^{(1)}(1/2)$. By Lemma 4, we see the Euler numbers E_n are given by the weighted lattice paths $(1, 0, -k^2)$, which means that the horizontal paths are eliminated. Therefore, E_n counts the weighted *Dyck paths*, related to Catalan numbers C_n [15, Ex. 25]. For example, when $n = 6$, there are

$$C_3 := \frac{1}{4}\binom{6}{3} = 5$$

weighted Dyck paths, listed as follows:



Then, by noting that each diagonally down path from (j,k) to $(j+1,k-1)$ has weight $-k^2$, we have

$$-61 = E_6 = (-1)^3\left(3^2 2^2 1^2 + 2^2 2^2 1^2 + 1^2 2^2 1^2 + 2^2 1^2 1^2 + 1^2 1^2 1^2\right).$$

Remark. This reconfirms that Euler numbers are integers, odd index terms vanish, and even terms have alternating signs. (See also [13, Entries 24.2.7 and 24.2.9].)

4. Analogue to Bernoulli polynomials

The *higher-order Bernoulli polynomials* $B_n^{(p)}(x)$ are defined by

$$\left(\frac{z}{e^z-1}\right)^p e^{zx} = \sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{z^n}{n!}.$$

When $p = 1$, $B_n^{(1)}(x) = B_n(x)$ and $B_n(0) = B_n$. Probabilistic interpretation for $B_n(x)$ can be found, e.g., [5, eq. 2.14]. Touchard [16, eq. 44] computed the orthogonal polynomials with respect to Bernoulli numbers, denoted by $R_n(y)$. More specifically, $R_0(y) = 1$, $R_1(y) = y + 1/2$ and for $n \geq 1$,

$$R_{n+1}(y) = \left(y+\frac{1}{2}\right)R_n(y) - \frac{n^4}{4(2n+1)(2n-1)}R_{n-1}(y).$$

Following similar steps, we shall obtain analogues of Theorem 1 and Theorem 3 for Bernoulli polynomials. The proof is omitted.

Theorem 17. *Let $\varrho_n(y)$ be the orthogonal polynomials with respect to $B_n(x)$, i.e., for integers r and n, with $0 \leq r < n$,*

$$y^r \varrho_n(y)|_{y^k=B_k(x)} = 0.$$

Then, $\varrho_0(y) = 1$, $\varrho_1(y) = y - x + 1/2$ and for $n \geq 1$,

$$\varrho_{n+1}(y) = \left(y - x + \frac{1}{2}\right)\varrho_n(y) - \frac{n^4}{4(2n+1)(2n-1)}\varrho_{n-1}(y). \tag{4.1}$$

In particular,

$$\varrho_n(y) = \frac{n!}{(n+1)_n}p_n\left(y;\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),$$

where $p_n(y;a,b,c,d)$ is the continuous Hahn polynomial [10, pp. 200–202]. *Moreover, let $\mathfrak{B}_{n,k}$ be the generalized Motzkin numbers with special choice $\sigma_k = x - 1/2$ and $\tau_k = -k^4/(4(2k+1)(2k-1))$. Then $\mathfrak{B}_{n,0} = B_n(x)$. The matrix*

$$RB := \begin{pmatrix} x-\frac{1}{2} & -\frac{1}{12} & 0 & 0 & \cdots & 0 & \cdots \\ 1 & x-\frac{1}{2} & -\frac{4}{15} & 0 & \cdots & 0 & \cdots \\ 0 & 1 & x-\frac{1}{2} & \ddots & \ddots & \vdots & \cdots \\ 0 & 0 & 1 & \ddots & -\frac{n^4}{4(2n+1)(2n-1)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & x-\frac{1}{2} & -\frac{(n+1)^4}{4(2n+1)(2n+3)} & \cdots \\ 0 & 0 & 0 & \ddots & 1 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \ddots \end{pmatrix}$$

generate all $B_n(x)$ through the power of its left upper block, as an analogue to Theorem 14.

Remark. We also tried to consider the analogue on $B_n^{(p)}(x)$. However, the rational coefficients $n^4/(4(2n+1)(2n-1))$ provides more difficulties than n^2 that appears in (1.5). Moreover, the convolution property (2.8) for Meixner-Pollaczek polynomials fails for continuous Hahn polynomials. Therefore, we only have the following computation and conjectures.

Let $\varrho_{n+1}^{(p)}(y)$ be the monic orthogonal polynomial with respect to $B_n^{(p)}(x)$, and assume the three-term recurrence is

$$\varrho_{n+1}^{(p)}(y) = \left(y - a_n^{(p)}\right)\varrho_n^{(p)}(y) - b_n^{(p)}\varrho_{n-1}(y).$$

To compute $\varrho_n^{(p)}(y)$, one could use, e.g., [8, eq. 2.1.10], which does not give explicit formulas for $a_n^{(p)}$ and $b_n^{(p)}$. However, for $a_n^{(p)}$, Lemma 4 implies that

$$a_n^{(p)} = x - p/2.$$

The first several terms of $b_n^{(p)}$ is given in the following table

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
$n=1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$
$n=2$	$\frac{4}{15}$	$\frac{13}{30}$	$\frac{3}{5}$	$\frac{23}{30}$	$\frac{14}{15}$
$n=3$	$\frac{81}{140}$	$\frac{372}{455}$	$\frac{1339}{1260}$	$\frac{2109}{1610}$	$\frac{1527}{980}$
$n=4$	$\frac{64}{63}$	$\frac{3736}{2821}$	$\frac{138688}{84357}$	$\frac{668543}{339549}$	$\frac{171830}{74823}$
$n=5$	$\frac{625}{396}$	$\frac{1245075}{636988}$	$\frac{299594775}{127670972}$	$\frac{42601023200}{15509529057}$	$\frac{3638564965}{1154491404}$

TABLE 2. $b_n^{(p)}$ for $1 \le n,\ p \le 5$

Here, the first column is $b_n^{(1)} = n^4/(4(2n+1)(2n-1))$, as that in the last term of (4.1). Also, one can easily see that the first row is linear as $b_1^{(p)} = p/12$, so is the second row $b_2^{(p)} = (5p+3)/10$. The following conjecture is due to Karl Dilcher.

Conjecture 18. *The third row is given by*

$$b_3^{(p)} = \frac{175p^2 + 315p + 158}{140(2p+3)};$$

the fourth row satisfies

$$b_4^{(p)} = \frac{6125p^4 + 25725p^3 + 41965p^2 + 29547p + 7230}{21(5p+3)(175p^2 + 315p + 158)};$$

and the fifth row is

$b_5^{(p)} = 25(5p+3)(471625p^6+3678675p^5+12324235p^4+22096305p^3+22009540p^2+11549748p+2519472)\Big/(132(175p^2+315p+158)(6125p^4+25725p^3+41965p^2+29547p+7230))$.

Remark. We do not have a conjecture on the general formula for $b_n^{(p)}$.

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