MULTIPLICITIES OF EIGENVALUES OF THE DIFFUSION OPERATOR WITH RANDOM JUMPS FROM THE BOUNDARY

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ABSTRACT. This paper deals with a non-self-adjoint differential operator which is associated with a diffusion process with random jumps from the boundary. Our main result is that the algebraic multiplicity of an eigenvalue is equal to its order as a zero of the characteristic function $\Delta(\lambda)$. This is a new criterion for determining the multiplicities of eigenvalues for concrete operators.

1. Introduction

This article investigates the non-self-adjoint differential operator L in $L^2_w(J,\mathbb{C})$ generated by the differential expression

$$Ly = ly := b_0(x)y'' + b_1(x)y'$$

and

$$\mathrm{dom}(L) := \left\{ y \in L^2_w(J,\mathbb{C}) \;\middle|\; \begin{array}{l} y, y' \in AC[0,1], Ly \in L^2_w(J,\mathbb{C}) \\ y(0) = \int_0^1 y(x)\mathrm{d}\nu_0(x),\ y(1) = \int_0^1 y(x)\mathrm{d}\nu_1(x) \end{array} \right\}.$$

Here ν_0, ν_1 are probability distributions on $J := (0,1)$ and

$$w := -\frac{1}{b_0},\ \frac{b_1}{b_0} \in L^1(J,\mathbb{R}),\ b_0 < 0 \text{ a.e. on } (0,1).$$

It is well known that the operator L is associated with a diffusion process with jumping boundary, which has attracted enormous interest in the last decades for various probability considerations and practical interests in genetics (see, e.g., [1–10] and the references therein). In this process, whenever the boundary of the interval $[0,1]$ is reached, the diffusion gets redistributed in $(0,1)$ according to the probability distributions ν_0, ν_1, runs again until it hits the boundary, gets redistributed and repeats this behavior forever. It should be noted that the above type of nonlocal boundary conditions can be already found in the fundamental work [7] of W. Feller, which characterized completely the analytic structure of one-dimensional diffusion processes.

We mention that this family of non-self-adjoint differential operators has quite interesting spectral-theoretic properties (see, e.g., [1, 5, 6, 8, 9] and the references therein). Particularly, in the case of $b_0(x) \equiv -1$, $b_1(x) \equiv 0$, Y. J. Leung, W. V. Li and Rakesh [6] discovered that the whole spectrum is real despite the fact that the operator L is non-self-adjoint. In addition, in this case the spectral gap is bounded

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2010 *Mathematics Subject Classification.* Primary 34L15; Secondary 47A10, 60J60.

Key words and phrases. diffusions, eigenvalues, non-self-adjoint, multiplicity.

*The work was done at the University of Vienna while the first author was visiting the Fakultät für Mathematik, supported by the China Scholarship Council. This research was also supported by the National Natural Science Foundation of China (Grant No. 11601372); the Science and Technology Research Project of Higher Education in Hebei Province (Grant No. QN2017044).

between the lowest and the second Dirichlet eigenvalue. Recently, M. Kolb and D. Krejčiřík in [8] analyzed the geometric and algebraic multiplicities of the eigenvalues from a purely operator-theoretic perspective in the case of $b_0(x) \equiv -1$, $b_1(x) \equiv 0$, $\nu_0 = \nu_1 = \delta_a$, $a \in (0,1)$, and showed that all the eigenvalues of L are algebraically simple if, and only if, $a \notin \mathbb{Q}$. Based on this, they studied the basis properties of L. One notes that many of the papers in question assume that the coefficients of L are constant (and sometimes in addition that $b_1 = 0$), and/or that the measures ν_0, ν_1 coincide and/or that these measures are degenerate measures. In this paper, there are no such assumptions at all and an interesting result on the multiplicities of eigenvalues of L is developed.

Let $y_1(x,\lambda)$ and $y_2(x,\lambda)$ be the fundamental solutions of

$$b_0(x)y''(x) + b_1(x)y'(x) = \lambda y(x) \tag{1.1}$$

determined by the initial conditions

$$y_1(0,\lambda) = y_2'(0,\lambda) = 1, y_1'(0,\lambda) = y_2(0,\lambda) = 0.$$

Denote

(1.2)

$$\Delta(\lambda) := \det \begin{pmatrix} \int_0^1 y_1(x,\lambda)\mathrm{d}\nu_0(x) - 1 & \int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0(x) \\ \int_0^1 y_1(x,\lambda)\mathrm{d}\nu_1(x) - y_1(1,\lambda) & \int_0^1 y_2(x,\lambda)\mathrm{d}\nu_1(x) - y_2(1,\lambda) \end{pmatrix}.$$

Then direct calculation yields that λ is an eigenvalue of L if and only if $\Delta(\lambda) = 0$.

Let us now present the main theorem of this paper.

Theorem 1.1. *Assume λ_0 be an eigenvalue of L with algebraic multiplicity $\chi(\lambda_0)$. Let n_0 denote the order of λ_0 as a zero of $\Delta(\lambda)$. Then $\chi(\lambda_0) = n_0$.*

We would like to emphasize that this theorem is useful for identifying the multiplicities of eigenvalues of the operator L. For example, it provides a straightforward method to obtain a main result in [8, Theorem 1] (see Remark 3.2). Moreover, suppose $b_0 < 0$ and $b_1 \neq 0$ be constants and $\nu_0 = \nu_1 = \delta_{\frac{1}{2}}$, then as a consequence of Theorem 1.1, Remark 3.3 shows us that all the eigenvalues of L are algebraically simple. This partially answers the last open problem listed in [8, Section 8].

2. Basic Properties and Preliminaries

Let us first recall some notations and definitions.

Notation 2.1. Let T be a linear operator in a Hilbert space H. In what follows, $\mathrm{dom}(T)$, $\ker(T)$ are the domain, the kernel of T, respectively; $\sigma(T)$, $\sigma_p(T)$, $\rho(T)$, denote the spectrum, point spectrum, the resolvent set of T, respectively; $R_\lambda(T) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, is the resolvent of T.

Definition 2.2. Let T be a linear operator in a Hilbert space H. The smallest integer $p > 0$ such that $\ker(T^p) = \ker(T^{p+1})$ is called the ascent of T and it is denoted by $\alpha(T)$.

Definition 2.3. Let T be a closed linear operator in a Hilbert space H and let λ_0 be an eigenvalue of T. Then the space $\ker((T - \lambda_0 I))$ is called the eigenspace of T corresponding to λ_0, and its dimension is called the **geometric multiplicity** of λ_0. The space $\bigcup_{n=1}^{\infty} \ker((T - \lambda_0 I)^n)$ is called the generalized eigenspace of T corresponding to λ_0, with its dimension referred to as the **algebraic multiplicity** of λ_0.

In this section, we mainly prove the following proposition which will be used in the proof of our main theorem.

Proposition 2.4. The operator L is closed and has a purely discrete spectrum. Moreover, for any point $\lambda_0 \in \sigma(L)$, $\alpha(L-\lambda_0 I)$ is finite.

In order to prove Proposition 2.4, we first consider the differential operator L_0 in $L^2_w(J,\mathbb{C})$ defined by

$$\begin{aligned} L_0 y : &= b_0(x)y'' + b_1(x)y', \\ \mathrm{dom}(L_0) : &= \left\{ y \in L^2_w(J,\mathbb{C}) \;\middle|\; \begin{array}{l} y, y' \in AC[0,1], L_0 y \in L^2_w(J,\mathbb{C}), \\ y(0) = y(1) = 0 \end{array} \right\}. \end{aligned}$$

It is well known that

$$(R_\lambda(L_0)f)(x) = \int_0^1 G^0_\lambda(x,t)f(t)\mathrm{d}t,\ x \in [0,1]\,,\ f \in L^2_w(J,\mathbb{C})$$

where

$$G^0_\lambda(x,t) = \begin{cases} \frac{y_2(t,\lambda)[y_2(x,\lambda)y_1(1,\lambda)-y_1(x,\lambda)y_2(1,\lambda)]}{b_0(t)W(t)y_2(1,\lambda)}, 0 \le t \le x, \\ \frac{y_2(x,\lambda)[y_2(t,\lambda)y_1(1,\lambda)-y_1(t,\lambda)y_2(1,\lambda)]}{b_0(t)W(t)y_2(1,\lambda)}, x \le t \le 1. \end{cases}$$

Here $W(x) = \exp(-\int_0^x \frac{b_1(t)}{b_0(t)}\mathrm{d}t)$ is the Wronskian of y_1 and y_2. It is obvious that $R_\lambda(L_0) : L^2_w(J,\mathbb{C}) \longrightarrow \mathrm{dom}(L_0)$ is a compact operator for $\lambda \in \rho(L_0) = \mathbb{C}\backslash\sigma(L_0)$, where $\sigma(L_0) = \{\lambda_n\}$ and λ_n are zeros of the entire function $y_2(1,\lambda)$([12, Sec. III., Example 6.11]).

Next, we give the formula on the resolvent $R_\lambda(L)$, following which Proposition 2.4 can be proved directly.

Lemma 2.5. *For every $\lambda \in \mathbb{C}\backslash\left[\sigma(L_0)\cup\sigma_p(L)\right]$, the resolvent $R_\lambda(L)$ of L admits the following decomposition*

$$\begin{aligned} (R_\lambda(L)f)(x) &= (R_\lambda(L_0)f)(x) + g_0(x)\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_0(x) \\ &\quad + g_1(x)\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_1(x) \end{aligned} \tag{2.1}$$

for each $f \in L^2_w(J,\mathbb{C})$ and $x \in [0,1]$, where

$$g_0(x) = \frac{\left[y_2(1,\lambda) - \int_0^1 y_2(x,\lambda)\mathrm{d}\nu_1(x)\right]y_1(x,\lambda) - \left[y_1(1,\lambda) - \int_0^1 y_1(x,\lambda)\mathrm{d}\nu_1(x)\right]y_2(x,\lambda)}{\Delta(\lambda)}$$

and

$$g_1(x) = \frac{\left[1 - \int_0^1 y_1(x,\lambda)\mathrm{d}\nu_0(x)\right]y_2(x,\lambda) + y_1(x,\lambda)\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0(x)}{\Delta(\lambda)}.$$

Proof. Firstly, it is easy to see that $R_\lambda(L)$ is a bounded operator on $L^2_w(J,\mathbb{C})$. In fact, the last two terms of the decomposition represent finite rank perturbations of the compact operator $R_\lambda(L_0)$. More specifically, for $i = 0, 1$,

$$g_i(x)\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_i(x) = g_i(x)\int_0^1\int_0^1 G^0_\lambda(x,t)f(t)\mathrm{d}t\mathrm{d}\nu_i(x)$$

are continuous on $[0,1]$ for $\lambda \in \mathbb{C}\backslash\left[\sigma(L_0)\cup\sigma_p(L)\right]$.

Next, we prove that $R_\lambda(L)f \in \mathrm{dom}(L)$. Indeed, the fact

$$(R_\lambda(L_0)f)(0) = (R_\lambda(L_0)f)(1) = 0$$

yields that

$$\begin{aligned}(R_\lambda(L)f)(0) &= \int_0^1 (R_\lambda(L)f)(x)\mathrm{d}\nu_0(x)\\ &= \frac{y_2(1,\lambda) - \int_0^1 y_2(x,\lambda)\mathrm{d}\nu_1(x)}{\Delta(\lambda)}\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_0(x)\\ &\quad+\frac{\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0(x)}{\Delta(\lambda)}\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_1(x)\end{aligned}$$

and

$$\begin{aligned}&(R_\lambda(L)f)(1)\\ =&\int_0^1 (R_\lambda(L)f)(x)\mathrm{d}\nu_1(x)\\ =&\frac{y_2(1,\lambda)\int_0^1 y_1(x,\lambda)\mathrm{d}\nu_1(x) - y_1(1,\lambda)\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_1(x)}{\Delta(\lambda)}\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_0(x)\\ &+\frac{\left[1-\int_0^1 y_1(x,\lambda)\mathrm{d}\nu_0(x)\right]y_2(1,\lambda) + y_1(1,\lambda)\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0(x)}{\Delta(\lambda)}\int_0^1 (R_\lambda(L_0)f)(x)\mathrm{d}\nu_1(x).\end{aligned}$$

Moreover, it is easy to deduce that

$$b_0(x)(R_\lambda(L)f)'' + b_1(x)(R_\lambda(L)f)' - \lambda(R_\lambda(L)f) = f \in L^2_w(J,\mathbb{C}).$$

Therefore, $R_\lambda(L)$ is a bounded operator from $L^2_w(J,\mathbb{C})$ to $\mathrm{dom}(L)$ and $R_\lambda(L)$ is the right inverse of $L-\lambda$. It remains to show that $R_\lambda(L)$ is the left inverse of $L-\lambda$. In fact, for every $\Psi \in \mathrm{dom}(L)$, denote

$$\Psi_0(x) = \left[R_\lambda(L_0)(L-\lambda)\Psi\right](x),\ \ \widetilde{\Psi} := \Psi_0 - \Psi.$$

Then

$$\begin{cases} b_0(x)\widetilde{\Psi}'' + b_1(x)\widetilde{\Psi}' = \lambda\widetilde{\Psi},\\ \widetilde{\Psi}(0) = \int_0^1 \widetilde{\Psi}(x)\mathrm{d}\nu_0(x) - \int_0^1 \Psi_0(x)\mathrm{d}\nu_0(x),\\ \widetilde{\Psi}(1) = \int_0^1 \widetilde{\Psi}(x)\mathrm{d}\nu_1(x) - \int_0^1 \Psi_0(x)\mathrm{d}\nu_1(x).\end{cases}$$

This yields that

$$\widetilde{\Psi}(x) = -g_0(x)\int_0^1 \Psi_0(x)\mathrm{d}\nu_0(x) - g_1(x)\int_0^1 \Psi_0(x)\mathrm{d}\nu_1(x).$$

Thus for every $\Psi \in \mathrm{dom}(L)$ and $\lambda \in \mathbb{C}\backslash\left[\sigma(L_0)\cup\sigma_p(L)\right]$, it follows from (2.1) that

$$\begin{aligned}&\left[R_\lambda(L)(L-\lambda)\Psi\right](x)\\ =&\ \Psi_0(x) + g_0(x)\int_0^1 \left[R_\lambda(L_0)(L-\lambda)\Psi\right](x)\mathrm{d}\nu_0(x)\\ &+g_1(x)\int_0^1 \left[R_\lambda(L_0)(L-\lambda)\Psi\right](x)\mathrm{d}\nu_1(x)\\ =&\ \Psi_0(x) - \widetilde{\Psi}(x) = \Psi(x).\end{aligned}$$

This completes the proof. □

Remark 2.6. We mention here that in [2, 10], the compactness of the semigroup generated by the operator L have already been studied even in higher dimensions. However, there more stringent assumptions on the coefficients are needed and the authors prefer often to work in different spaces.

In addition, let us recall several basic facts.

Lemma 2.7. *The initial problem consisting of equation* (1.1) *and the initial conditions*

$$y(0,\lambda) = h,\ \ y'(0,\lambda) = k, \tag{2.2}$$

where $h,\, k \in \mathbb{C}$*, has a unique solution* $y(x,\lambda)$*. And each of the functions* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *is continuous on* $[0,1]\times\mathbb{C}$*, in particular, the functions* $y(x,\lambda)$ *and* $y'(x,\lambda)$ *are entire functions of* $\lambda \in \mathbb{C}$*.*

Proof. See [13]. □

Remark 2.8. In fact, from [13] one also has the derivative of $y(x,\lambda)$ with respect to λ is given by

$$y'_\lambda(x,\lambda) = \int_0^x \frac{y_2(x,\lambda)y_1(t,\lambda) - y_1(x,\lambda)y_2(t,\lambda)}{b_0(t)\exp(-\int_0^t \frac{b_1(s)}{b_0(s)}\mathrm{d}s)} y(t,\lambda)\mathrm{d}t.$$

Remark 2.9. Lemma 2.7 implies that $\Delta(\lambda)$ is an entire function of $\lambda \in \mathbb{C}$.

Remark 2.10. Consider the differential operator $\widetilde{L}_0$ in $L^2_w(J,\mathbb{C})$ defined by

$$\begin{aligned} \widetilde{L}_0 y \ &: = b_0(x)y'' + b_1(x)y', \\ \mathrm{dom}(\widetilde{L}_0) \ &: = \left\{ y \in L^2_w(J,\mathbb{C}) \;\middle|\; \begin{array}{l} y, y' \in AC[0,1], \widetilde{L}_0 y \in L^2_w(J,\mathbb{C}), \\ y(0) = y'(0) = 0 \end{array} \right\}. \end{aligned}$$

It is obvious that the resolvent set $\rho(\widetilde{L}_0) = \mathbb{C}$. Moreover, direct calculation yields that for each $f \in L^2_w(J,\mathbb{C})$ and $x \in [0,1]$,

$$(R_\lambda(\widetilde{L}_0)f)(x) = \int_0^x \frac{y_2(x,\lambda)y_1(t,\lambda) - y_1(x,\lambda)y_2(t,\lambda)}{b_0(t)\exp(-\int_0^t \frac{b_1(s)}{b_0(s)}\mathrm{d}s)} f(t)\mathrm{d}t.$$

Therefore, $(R_\lambda(\widetilde{L}_0)f)(x)$ is an entire function of $\lambda \in \mathbb{C}$.

Now we are in a position to prove Proposition 2.4.

Proof of Proposition 2.4. From Lemma 2.5, it follows that the resolvent $R_\lambda(L)$ of L is a compact operator, thus the operator L is closed and its spectrum is purely discrete. Moreover, for any point $\lambda_0 \in \sigma(L)$, from Lemma 2.5 we know that λ_0 is a pole of $R_\lambda(L)$ of finite order. Thus the finiteness of $\alpha(L-\lambda_0 I)$ follows from [14, Chap. V, Theorem 10.1]. This proves Proposition 2.4. □

3. Proof of Theorem 1.1 and Remarks

Based on the statements given in the previous section, we present the proof of Theorem 1.1 in this section and use this result to solve several problems.

Proof of Theorem 1.1. Let m_0 denote the ascent of the operator $L - \lambda_0 I$, then $\chi(\lambda_0) = \dim\ker((L-\lambda_0 I)^{m_0})$. Denote the geometric multiplicity of the eigenvalue λ_0 by m. Then it is obvious that $m \leq 2$. Note that we will mainly prove the statement of this theorem in the case of $m = 1$, since the proof for $m = 2$ can be given only with a slight modification.

When $m = 1$, the proof can be divided into two steps.

Step 1. When λ is sufficiently close to λ_0, we first construct two linear independent solutions $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ of the equation $(l-\lambda I)y = 0$ via the generalized eigenfunctions of λ_0. Let us recall that $ly = b_0(x)y'' + b_1(x)y'$.

Define a linear operator F on the finite dimensional space $\ker((L - \lambda_0 I)^{m_0})$ as follows:

$$F = (L - \lambda_0 I)|\ker((L - \lambda_0 I)^{m_0}).$$

Then $F^{m_0} = 0$ and $F^{m_0-1} \neq 0$, i.e., F is nilpotent with index m_0. It follows from [15, Chapter 57, Theorem 2] that there exist functions

$$\eta, F\eta, \dots F^{m_0-1}\eta$$

form a basis of the generalized space $\ker((L - \lambda_0 I)^{m_0})$. Note that in this case $m_0 = \chi(\lambda_0)$. Then denote

$$\xi_{0,1} := F^{m_0-1}\eta,\ \xi_{1,1} := F^{m_0-2}\eta, \cdots\ \xi_{m_0-1,1} := \eta.$$

Select another solution $\xi_{0,2}$ of the equation $(l - \lambda_0 I)y = 0$ such that $\xi_{0,1}$ and $\xi_{0,2}$ are fundamental solutions of $(l - \lambda_0 I)y = 0$.

For $\lambda \in \mathbb{C}$, define

$$\phi_1(x, \lambda) = \sum_{k=0}^{m_0-1} (\lambda - \lambda_0)^k \xi_{k,1}(x) + (\lambda - \lambda_0)^{m_0} (R_\lambda(\widetilde{L}_0)\eta)(x), \tag{3.1}$$

$$\phi_2(x, \lambda) = \xi_{0,2}(x) + (\lambda - \lambda_0)(R_\lambda(\widetilde{L}_0)\xi_{0,2})(x). \tag{3.2}$$

Note that $\phi_i(x, \lambda_0) = \xi_{0,i}(x)$, $i = 1, 2$. Then we will show that $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ are linear independent solutions of the equation $(l - \lambda I)y = 0$. In fact,

$$\begin{aligned}
&((l - \lambda I)\phi_1)(x, \lambda) \\
= &\sum_{k=1}^{m_0-1} (\lambda - \lambda_0)^k ((l - \lambda_0 I)\xi_{k,1})(x) - \sum_{k=0}^{m_0-1} (\lambda - \lambda_0)^{k+1} \xi_{k,1}(x) + (\lambda - \lambda_0)^{m_0} \eta(x) \\
= &\sum_{k=1}^{m_0-1} (\lambda - \lambda_0)^k \xi_{k-1,1}(x) - \sum_{k=0}^{m_0-1} (\lambda - \lambda_0)^{k+1} \xi_{k,1}(x) + (\lambda - \lambda_0)^{m_0} \xi_{m_0-1,1}(x) \\
= &\ 0
\end{aligned}$$

and

$$((l - \lambda I)\phi_2)(x, \lambda) = ((l - \lambda I)\xi_{0,2})(x) + (\lambda - \lambda_0)\xi_{0,2}(x) = 0.$$

Moreover, since $\xi_{0,1}$ and $\xi_{0,2}$ are linear independent solutions of $(l - \lambda_0 I)y = 0$, we have

$$\det \begin{pmatrix} \phi_1(0, \lambda_0) & \phi_2(0, \lambda_0) \\ \phi_1'(0, \lambda_0) & \phi_2'(0, \lambda_0) \end{pmatrix} = \det \begin{pmatrix} \xi_{0,1}(0) & \xi_{0,2}(0) \\ \xi_{0,1}'(0) & \xi_{0,2}'(0) \end{pmatrix} \neq 0.$$

As a consequence of Remark 2.10, $\phi_i(0, \lambda)$ and $\phi_i'(0, \lambda)$, $i = 1, 2$ are entire functions of $\lambda \in \mathbb{C}$, hence there exists a number $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$, $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ are linear independent solutions of $(l - \lambda I)y = 0$.

Step 2. Based on step 1, when $|\lambda - \lambda_0| < \delta$, one has

$$\begin{pmatrix} y_1(x, \lambda) \\ y_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) \end{pmatrix} \begin{pmatrix} \phi_1(x, \lambda) \\ \phi_2(x, \lambda) \end{pmatrix} \tag{3.3}$$

and $\det\begin{pmatrix} b_{11}(\lambda) & b_{12}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) \end{pmatrix} \neq 0$. Thus from (3.3), the definition of ϕ_1, ϕ_2, and the fact $\xi_{k,1} \in \mathrm{dom}(L-\lambda_0 I)$, $k = 0, 1, \dots m_0 - 1$, it follows that

$$\begin{aligned} &\Delta(\lambda) \\ = \;& \det\begin{pmatrix} \int_0^1 y_1(x,\lambda)\mathrm{d}\nu_0(x) - y_1(0,\lambda) & \int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0(x) - y_2(0,\lambda) \\ \int_0^1 y_1(x,\lambda)\mathrm{d}\nu_1(x) - y_1(1,\lambda) & \int_0^1 y_2(x,\lambda)\mathrm{d}\nu_1(x) - y_2(1,\lambda) \end{pmatrix} \\ = \;& (\lambda-\lambda_0)^{m_0}\det(g_{i,j}(\lambda))\det\begin{pmatrix} b_{11}(\lambda) & b_{21}(\lambda) \\ b_{12}(\lambda) & b_{22}(\lambda) \end{pmatrix}, \end{aligned}$$

where $|\lambda-\lambda_0| < \delta$, and for $i = 1, 2$,

$$\begin{aligned} g_{i,1}(\lambda) &= \int_0^1 (R_\lambda(\widetilde{L}_0)\eta)(x)\mathrm{d}\nu_{i-1}(x) - (R_\lambda(\widetilde{L}_0)\eta)(i-1), \\ g_{i,2}(\lambda) &= (\lambda-\lambda_0)\left[\int_0^1 (R_\lambda(\widetilde{L}_0)\xi_{0,2})(x)\mathrm{d}\nu_{i-1}(x) - (R_\lambda(\widetilde{L}_0)\xi_{0,2})(i-1)\right] \\ &\quad + \int_0^1 \xi_{0,2}(x)\mathrm{d}\nu_{i-1}(x) - \xi_{0,2}(i-1). \end{aligned}$$

Recall that in this case $m_0 = \chi(\lambda_0)$. Thus in order to show the order of λ_0 as a zero of $\Delta(\lambda)$ is equal to $\chi(\lambda_0)$, it is sufficient to prove that $\det(g_{i,j}(\lambda_0)) \neq 0$ since $g_{i,j}(\lambda)$ are entire functions of $\lambda \in \mathbb{C}$. Otherwise, there exists a constant c such that

$$\begin{aligned} &\begin{pmatrix} \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta)(x)\mathrm{d}\nu_0(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta)(0) \\ \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta)(x)\mathrm{d}\nu_1(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta)(1) \end{pmatrix} \\ = \;& c\begin{pmatrix} \int_0^1 \xi_{0,2}(x)\mathrm{d}\nu_0(x) - \xi_{0,2}(0) \\ \int_0^1 \xi_{0,2}(x)\mathrm{d}\nu_1(x) - \xi_{0,2}(1) \end{pmatrix}. \end{aligned}$$

Denote $u(x) = (R_{\lambda_0}(\widetilde{L}_0)\eta)(x) - c\xi_{0,2}(x)$, then the above equation implies that

$$\int_0^1 u(x)\mathrm{d}\nu_0(x) = u(0) \text{ and } \int_0^1 u(x)\mathrm{d}\nu_1(x) = u(1).$$

Therefore,

$$(l-\lambda_0 I)u = \eta \in \ker((L-\lambda_0 I)^{m_0}) \tag{3.4}$$

and hence

$$u \in \ker((L-\lambda_0 I)^{m_0+1}) = \ker((L-\lambda_0 I)^{m_0}).$$

This implies that there exists constants α_i such that $u = \sum_{i=0}^{m_0-1} \alpha_i F^i\eta$. Thus

$$(l-\lambda_0 I)u = \sum_{i=0}^{m_0-1} \alpha_i (l-\lambda_0 I)F^i\eta = \sum_{i=0}^{m_0-2} \alpha_i F^{i+1}\eta = \sum_{i=1}^{m_0-1} \alpha_{i-1}F^i\eta.$$

This together with (3.4) yield that $\eta = \sum_{i=1}^{m_0-1} \alpha_{i-1}F^i\eta$ which contradicts the linear independence of $\eta, F\eta, \dots F^{m_0-1}\eta$. This proves $\det(g_{i,j}(\lambda_0)) \neq 0$. Hence the statement of Theorem 1.1 in the case of $m = 1$ is proved.

Now we turn to the case $m = 2$. We only need to make slight modifications on the solutions ϕ_1, ϕ_2 and $(g_{i,j}(\lambda))$. Note that it follows from [15, Chapter 57, Theorem 2] that there exists functions $\eta_{1,}$ $\eta_2 \in \ker((L-\lambda_0 I)^{m_0})$ such that

$$\begin{aligned} &\eta_1, F\eta_1, \dots\ F^{q_1-1}\eta_1, \\ &\eta_2, F\eta_2, \dots\ F^{q_2-1}\eta_2 \end{aligned}$$

form a basis of the generalized space $\ker((L-\lambda_0 I)^{m_0})$ where $q_1 + q_2 = \chi(\lambda_0)$, $m_0 = q_1 \geq q_2 > 0$ and $F^{q_1}\eta_1 = F^{q_2}\eta_2 = 0$. In this case, denote

$$\xi_{0,i} := F^{q_i-1}\eta_i,\ \xi_{1,i} := F^{q_i-2}\eta_i, \cdots\ \xi_{m_0-1,i} := \eta_i,\ i = 1,2.$$

Hence $\xi_{0,1}$ and $\xi_{0,2}$ are fundamental solutions of $(l-\lambda_0 I)y = 0$.

For $\lambda \in \mathbb{C}$, define

$$\phi_i(x,\lambda) = \sum_{k=0}^{q_i-1}(\lambda-\lambda_0)^k \xi_{k,i}(x) + (\lambda-\lambda_0)^{m_0}(R_\lambda(\widetilde{L}_0)\eta_i)(x),\ i = 1,2. \tag{3.5}$$

Note that $\phi_i(x,\lambda_0) = \xi_{0,i}(x)$, $i = 1,2$. By a process similar to that in the case $m = 1$, it can be obtained that

$$g_{i,j}(\lambda) = \int_0^1 (R_\lambda(\widetilde{L}_0)\eta_j)(x)\mathrm{d}\nu_{i-1}(x) - (R_\lambda(\widetilde{L}_0)\eta_j)(i-1).$$

Similarly, $\det(g_{i,j}(\lambda_0)) \neq 0$. Otherwise, there exists a constant c such that

$$\begin{aligned} &\begin{pmatrix} \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(x)\mathrm{d}\nu_0(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(0) \\ \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(x)\mathrm{d}\nu_1(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(1) \end{pmatrix} \\ = \quad c &\begin{pmatrix} \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(x)\mathrm{d}\nu_0(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(0) \\ \int_0^1 (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(x)\mathrm{d}\nu_1(x) - (R_{\lambda_0}(\widetilde{L}_0)\eta_2)(1) \end{pmatrix}. \end{aligned}$$

Denote $u(x) = (R_{\lambda_0}(\widetilde{L}_0)\eta_1)(x) - c(R_{\lambda_0}(\widetilde{L}_0)\eta_2)(x)$, then $\int_0^1 u(x)\mathrm{d}\nu_0(x) = u(0)$ and $\int_0^1 u(x)\mathrm{d}\nu_1(x) = u(1)$. Hence

$$(l-\lambda_0 I)u = \eta_1 - c\eta_2 \in \ker((L-\lambda_0 I)^{m_0})$$

and

$$u \in \ker((L-\lambda_0 I)^{m_0+1}) = \ker((L-\lambda_0 I)^{m_0}).$$

This implies that there exist constants α_i, β_i such that $u = \sum\limits_{i=0}^{q_1-1} \alpha_i F^i \eta_1 + \sum\limits_{i=0}^{q_2-1} \beta_i F^i \eta_2$. Hence

$$\eta_1 - c\eta_2 = (l-\lambda_0 I)u = \sum_{i=1}^{q_1-1} \alpha_{i-1} F^i \eta_1 + \sum_{i=1}^{q_2-1} \beta_{i-1} F^i \eta_2.$$

This contradicts the linear independence of $\eta_i, F\eta_i, \dots\ F^{q_i-1}\eta_i, i = 1,2$. Now the proof is completed. □

Remark 3.1. From the above proofs, one can find that Theorem 1.1 also holds even if $\frac{1}{b_0}$, $\frac{b_1}{b_0} \in L^1(J,\mathbb{C})$.

Based on Theorem 1.1, we conclude this paper with three remarks on two concrete eigenvalue problems which have been treated in [5, 8, 11].

Remark 3.2. Consider the eigenvalue problem with coefficients $b_0 \equiv -1$, $b_1 \equiv 0$ and $\nu_0 = \nu_1 = \delta_a$, $a \in (0,1)$, i.e.,

$$\begin{cases} -y''(x) = \lambda y(x), \ x \in (0,1), \\ y(0) = y(a) = y(1), \ a \in (0,1). \end{cases} \tag{3.6}$$

In [8, Theorem 1], M. Kolb and D. Krejčiřík showed all the eigenvalues of the problem (3.6) are algebraically simple if, and only if, $a \notin \mathbb{Q}$. Based on Theorem 1.1, this interesting result can be obtained from a different perspective.

In fact, from (1.2) we know that λ is an eigenvalue of the problem (3.6) if and only if

$$\Delta(\lambda) = -\frac{4}{\sqrt{\lambda}} \sin\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} = 0.$$

Furthermore,

$$\begin{aligned} \Delta'(\lambda) \;=\; & \frac{2}{\lambda^{\frac{3}{2}}} \sin\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} - \frac{1-a}{\lambda} \cos\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} \\ & -\frac{a}{\lambda} \sin\frac{\sqrt{\lambda}(1-a)}{2} \cos\frac{\sqrt{\lambda}a}{2} \sin\frac{\sqrt{\lambda}}{2} - \frac{1}{\lambda} \sin\frac{\sqrt{\lambda}(1-a)}{2} \sin\frac{\sqrt{\lambda}a}{2} \cos\frac{\sqrt{\lambda}}{2}. \end{aligned}$$

Thus it is easy to see that for any $\widehat{\lambda} \neq 0$, $\Delta(\widehat{\lambda}) = \Delta'(\widehat{\lambda}) = 0$ if and only if

$$\sin\frac{\sqrt{\widehat{\lambda}}(1-a)}{2} = \sin\frac{\sqrt{\widehat{\lambda}}a}{2} = \sin\frac{\sqrt{\widehat{\lambda}}}{2} = 0,$$

i.e.,

$$\widehat{\lambda} = (2m\pi)^2 = (\frac{2n\pi}{a})^2 = (\frac{2l\pi}{1-a})^2, \ m, \ n, \ l \in \mathbb{N} := \{1, 2, \ldots\}. \tag{3.7}$$

Hence for each $\widehat{\lambda} \neq 0$ which satisfies $\Delta(\widehat{\lambda}) = \Delta'(\widehat{\lambda}) = 0$, direct calculation yields that $\Delta''(\widehat{\lambda}) = 0$ and $\Delta'''(\widehat{\lambda}) = \frac{3a^2-3a}{8\widehat{\lambda}^2} \neq 0$. Obviously, $\Delta'(0) = \frac{a(a-1)}{2} \neq 0$. Thus it follows from Theorem 1.1 that the algebraic multiplicity of each eigenvalue of the problem (3.6) is either one or three. Moreover, since (3.7) implies that $a = \frac{n}{m} = 1 - \frac{l}{m} \in \mathbb{Q}$, we can easily conclude that all the eigenvalues of the problem (3.6) are algebraically simple if, and only if, $a \notin \mathbb{Q}$.

Remark 3.3. Consider the eigenvalue problem with constant coefficients $b_0 < 0$, $b_1 \in \mathbb{R}$ and $\nu_0 = \nu_1 = \delta_{\frac{1}{2}}$, i.e.,

$$\begin{cases} b_0 y''(x) + b_1 y'(x) = \lambda y(x), \ x \in (0,1), \\ y(0) = y(\frac{1}{2}) = y(1). \end{cases} \tag{3.8}$$

Assume $b_1 \neq 0$. It follows from Theorem 1.1 that each eigenvalue of the problem (3.8) is algebraically and geometrically simple. This partially answers the last open question posed by M. Kolb and D. Krejčiřík [8, Section 8].

In fact, under the transformation $v(x) = \exp(\frac{b_1 x}{2b_0}) y(x)$, problem (3.8) is equivalent to the following eigenvalue problem

$$\begin{cases} -v''(x) + qv(x) = -\frac{\lambda}{b_0} v(x), \ x \in (0,1), \\ v(0) = Av(\frac{1}{2}) = \ A^2 v(1) \end{cases} \tag{3.9}$$

where $q=\frac{1}{4}(\frac{b_1}{b_0})^2$ and $A=\exp(\frac{b_1}{-4b_0})$. Let $v_1(x,\lambda)$ and $v_2(x,\lambda)$ be the fundamental solutions of the differential equation in (3.9) with the initial conditions

$$v_1(0,\lambda)=v_2'(0,\lambda)=1,\ v_2(0,\lambda)=v_1'(0,\lambda)=0,\ \lambda\in\mathbb{C}. \tag{3.10}$$

Then $v_1(x,\lambda)=\cos\left(\sqrt{-\frac{\lambda}{b_0}-q}x\right)$, $v_2(x,\lambda)=\frac{\sin\left(\sqrt{-\frac{\lambda}{b_0}-q}x\right)}{\sqrt{-\frac{\lambda}{b_0}-q}}$. It can be easily obtained that λ is an eigenvalue of the problem (3.8) or (3.9) if and only if

$$\Delta_1(\lambda)=\det\begin{pmatrix} Av_1(\frac{1}{2},\lambda)-1 & Av_2(\frac{1}{2},\lambda)\\ 1-A^2v_1(1,\lambda) & -A^2v_2(1,\lambda)\end{pmatrix}=0.$$

For simplicity, let $u=-\frac{\lambda}{b_0}-q$, then denote

$$\widetilde{\Delta}_1(u):=\Delta_1(-b_0\,(u+q))=-2A^2\frac{\sin\frac{\sqrt{u}}{2}}{\sqrt{u}}\left(\frac{A^2+1}{2A}-\cos\frac{\sqrt{u}}{2}\right). \tag{3.11}$$

When $b_1\neq 0$, it is obvious that $\frac{A^2+1}{2A}>1$. Let u_n be the zeros of $\Delta_1(u)$. Then direct calculation yields that $\{u_n\}=\{u_{n,1}\}\cup\{u_{n,2}\}\cup\{u_{n,3}\}$,

$$\begin{aligned} u_{n,1} &= (2n\pi)^2,\ u_{n,2}=(4n\pi-2ir)^2,\ n\in\mathbb{N},\\ u_{n,3} &= (4n\pi+2ir)^2,\ n\in\mathbb{N}_0:=\{0,1,2,\ldots\}. \end{aligned}$$

Here $r>0$ and $\cosh r=\frac{A^2+1}{2A}$, i.e., $r=\frac{b_1}{-4b_0}$. Hence eigenvalues λ_n of the problem (3.8) or (3.9) are as follows: $\{\lambda_n\}=\{\lambda_{n,1}\}\cup\{\lambda_{n,2}\}\cup\{\lambda_{n,3}\}$,

$$\lambda_{n,1} = -4b_0n^2\pi^2-\frac{b_1^2}{4b_0},\ \lambda_{n,2}=-16b_0n^2\pi^2-2b_1n\pi i,\ n\in\mathbb{N}, \tag{3.12}$$

$$\lambda_{n,3} = -16b_0n^2\pi^2+2b_1n\pi i,\ n\in\mathbb{N}_0. \tag{3.13}$$

For each eigenvalue λ_n of the problem (3.8), one can easily obtain $\Delta_1'(\lambda_n)\neq 0$. In order to use Theorem 1.1 to show that each eigenvalue of the problem (3.8) is algebraically simple, it is sufficient to show that $\Delta_1(\lambda)\equiv\Delta(\lambda)$. Note that $\Delta(\lambda)$ is the characteristic function defined in (1.2). Denote

$$\widetilde{y}_1(x,\lambda):=\exp\left(-\frac{b_1x}{2b_0}\right)v_1(x,\lambda),\ \widetilde{y}_2(x,\lambda):=\exp\left(-\frac{b_1x}{2b_0}\right)v_2(x,\lambda),$$

then $\widetilde{y}_1(x,\lambda)$ and $\widetilde{y}_2(x,\lambda)$ are solutions of the differential equation in (3.8) determined by the initial conditions

$$\widetilde{y}_1(0,\lambda)=1,\ \widetilde{y}_1'(0,\lambda)=-\frac{b_1}{2b_0},\ \widetilde{y}_2(0,\lambda)=0,\ \ \widetilde{y}_2'(0,\lambda)=1,\lambda\in\mathbb{C}.$$

Thus $\widetilde{y}_1(x,\lambda)=y_1(x,\lambda)-\frac{b_1}{2b_0}y_2(x,\lambda)$, $\widetilde{y}_2(x,\lambda)=y_2(x,\lambda)$. Hence

$$\begin{aligned} \Delta_1(\lambda) &= \det\begin{pmatrix} \widetilde{y}_1(\frac{1}{2},\lambda)-1 & \widetilde{y}_2(\frac{1}{2},\lambda)\\ 1-\widetilde{y}_1(1,\lambda) & -\widetilde{y}_2(1,\lambda)\end{pmatrix}\\ &= \det\begin{pmatrix} y_1(\frac{1}{2},\lambda)-1 & y_2(\frac{1}{2},\lambda)\\ 1-y_1(1,\lambda) & -y_2(1,\lambda)\end{pmatrix}=\Delta(\lambda). \end{aligned}$$

Therefore, each eigenvalue of the problem (3.8) is algebraically and thus geometrically simple.

Remark 3.4. Denote the spectral gap of the problem (3.8) by $\gamma_1(\delta_{\frac{1}{2}})$, i.e.,

$$\gamma_1(\delta_{\frac{1}{2}}) := \inf\{\,\mathrm{Re}\lambda |\, \lambda \text{ is an eigenvalue of the problem (3.8) and } \lambda \neq 0\,\}.$$

Note that when $b_1 = 0$, $\lambda_n = -4b_0 n^2\pi^2$, $n \in \mathbb{N}_0$. This together with (3.12) and (3.13) yield

$$\gamma_1(\delta_{\frac{1}{2}}) = \begin{cases} -4b_0\pi^2 - \frac{b_1^2}{4b_0}, \text{ when } |b_1| \leq -4\sqrt{3}b_0\pi, \\ -16b_0\pi^2, \text{ when } |b_1| > -4\sqrt{3}b_0\pi, \end{cases}$$

which is already given in [5] and [11] by different approaches.

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