# THE THICKNESS OF $K_{1,n,n}$ AND $K_{2,n,n}$

XIA GUO AND YAN YANG

Abstract. The thickness of a graph $G$ is the minimum number of planar subgraphs whose union is $G$. In this paper, we obtain the thickness of complete 3-partite graph $K_{1,n,n}, K_{2,n,n}$ and complete 4-partite graph $K_{1,1,n,n}$.

## 1. Introduction

The *thickness* $\theta(G)$ of a graph $G$ is the minimum number of planar subgraphs whose union is $G$. It was first defined by W.T.Tutte [7] in 1963, then a few authors obtained the thickness of hypercubes [5], complete graphs [1, 2, 8] and complete bipartite graphs [3]. Naturally, people wonder about the thickness of the complete multipartite graphs.

A *complete $k$-partite graph* is a graph whose vertex set can be partitioned into $k$ parts, such that every edge has its ends in different parts and every two vertices in different parts are adjacent. Let $K_{p_1,p_2,\dots,p_k}$ denote a complete $k$-partite graph in which the $i$th part contains $p_i$ $(1 \le i \le k)$ vertices. For the complete 3-partite graph, Poranen proved $\theta(K_{n,n,n}) \le \lceil \frac{n}{2} \rceil$ in [6], then Yang [10] gave a new upper bound for $\theta(K_{n,n,n})$, i.e., $\theta(K_{n,n,n}) \le \lceil \frac{n+1}{3} \rceil + 1$ and obtained $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$, when $n \equiv 3 \pmod 6$. And also Yang [9] gave the thickness number of $K_{l,m,n}(l \le m \le n)$ when $l + m \le 5$ and showed that $\theta(K_{l,m,n}) = \lceil \frac{l+m}{2} \rceil$ when $l + m$ is even and $n > \frac{1}{2}(l + m - 2)^2$; or $l + m$ is odd and $n > (l + m - 2)(l + m - 1)$.

In this paper, we obtain the thickness of complete 3-partite graph $K_{1,n,n}$ and $K_{2,n,n}$, and we also deduce the thickness of complete 4-partite graph $K_{1,1,n,n}$ from that of $K_{2,n,n}$.

## 2. The thickness of $K_{1,n,n}$

In [3], Beineke, Harary and Moon gave the thickness of complete bipartite graphs $K_{m,n}$ for most value of $m$ and $n$, and their theorem implies the following result immediately.

**Lemma 2.1.** [3] *The thickness of the complete bipartite graph $K_{n,n}$ is*

$$\theta(K_{n,n}) = \lceil \frac{n+2}{4} \rceil.$$

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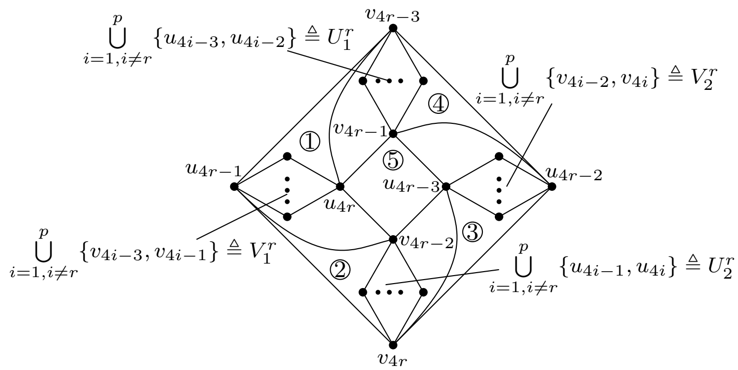
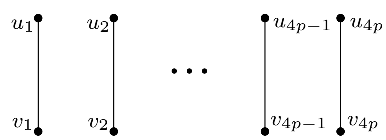
In [4], Chen and Yin gave a planar decomposition of the complete bipartite graph $K_{4p,4p}$ with $p+1$ planar subgraphs. Figure 1 shows their planar decomposition of $K_{4p,4p}$, in which $\{u_1, \ldots, u_{4p}\} = U$ and $\{v_1, \ldots, v_{4p}\} = V$ are the 2-partite vertex sets of it. Based on their decomposition, we give a planar decomposition of $K_{2,n,n}$ with $p+1$ subgraphs when $n \equiv 0$ or 3 (mod 4) and prove the following lemma.




(a) The graph $G_r$ $(1 \leq r \leq p)$




(b) The graph $G_{p+1}$

FIGURE 1. A planar decomposition of $K_{4p,4p}$

**Lemma 2.2.** *The thickness of the complete 3-partite graph $K_{1,n,n}$ and $K_{2,n,n}$ is*

$$\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = \lceil \frac{n+2}{4} \rceil,$$

*when $n \equiv 0$ or 3 (mod 4).*

*Proof.* Let the vertex partition of $K_{2,n,n}$ be $(X, U, V)$, where $X = \{x_1, x_2\}$, $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$.

When $n \equiv 0 (\text{mod } 4)$, let $n = 4p$ $(p \geq 1)$. Let $\{G_1, \ldots, G_{p+1}\}$ be the planar decomposition of $K_{n,n}$ constructed by Chen and Yin in [4]. As shown in Figure 1, the graph $G_{p+1}$ consists of $n$ paths of length one. We put all the $n$ paths in a row, place vertex $x_1$ on one side of the row and the vertex $x_2$ on the other side of the row, join both $x_1$ and $x_2$ to all vertices in $G_{p+1}$. Then we get a planar graph, denote it by $\widehat{G}_{p+1}$. It is easy to see that $\{G_1, \ldots, G_p, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,n,n}$. Therefore, we have $\theta(K_{2,n,n}) \leq p+1$. Since $K_{n,n} \subset K_{1,n,n} \subset K_{2,n,n}$, combining it with Lemma 2.1, we have

$$p+1 = \theta(K_{n,n}) \leq \theta(K_{1,n,n}) \leq \theta(K_{2,n,n}) \leq p+1,$$

that is, $\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = p+1$ when $n \equiv 0 (\text{mod } 4)$.

When $n \equiv 3 (\text{mod } 4)$, then $n = 4p + 3$ $(p \geq 0)$. When $p = 0$, from [9], we have $\theta(K_{1,3,3}) = \theta(K_{2,3,3}) = 2$. When $p \geq 1$, since $K_{n,n} \subset K_{1,n,n} \subset K_{2,n,n} \subset K_{2,n+1,n+1}$, according to Lemma 2.1 and $\theta(K_{2,4p,4p}) = p + 1$, we have

$$p + 2 = \theta(K_{n,n}) \leq \theta(K_{1,n,n}) \leq \theta(K_{2,n,n}) \leq \theta(K_{2,n+1,n+1}) = p + 2.$$

Then, we get $\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = p + 2$ when $n \equiv 3 (\text{mod } 4)$.

Summarizing the above, the lemma is obtained. □

**Lemma 2.3.** *There exists a planar decomposition of the complete 3-partite graph* $K_{1,4p+2,4p+2}$ $(p \geq 0)$ *with* $p + 1$ *subgraphs.*

*Proof.* Suppose the vertex partition of the complete 3-partite graph $K_{1,n,n}$ is $(X, U, V)$, where $X = \{x\}$, $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. When $n = 4p + 2$, we will construct a planar decomposition of $K_{1,4p+2,4p+2}$ with $p + 1$ planar subgraphs to complete the proof. Our construction is based on the planar decomposition $\{G_1, G_2, \ldots, G_{p+1}\}$ of $K_{4p,4p}$ given in [4], as shown in Figure 1 and the reader is referred to [4] for more details about this decomposition. For convenience, we denote the vertex set $\bigcup_{i=1, i\neq r}^{p} \{u_{4i-3}, u_{4i-2}\}$, $\bigcup_{i=1, i\neq r}^{p} \{u_{4i-1}, u_{4i}\}$, $\bigcup_{i=1, i\neq r}^{p} \{v_{4i-3}, v_{4i-1}\}$ and $\bigcup_{i=1, i\neq r}^{p} \{v_{4i-2}, v_{4i}\}$ by $U_1^r$, $U_2^r$, $V_1^r$ and $V_2^r$ respectively. We also label some faces of $G_r$ $(1 \leq r \leq p)$, as indicated in Figure 1, for example, the face 1 is bounded by $v_{4r-3}u_{4r}v_j u_{4r-1}$ in which $v_j$ is some vertex from $V_1^r$.

In the following, for $1 \leq r \leq p+1$, by adding vertices $x, u_{4p+1}, u_{4p+2}, v_{4p+1}, v_{4p+2}$ and some edges to $G_r$, and deleting some edges from $G_r$ such edges will be added to the graph $G_{p+1}$, we will get a new planar graph $\widehat{G}_r$ such that $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{1,4p+2,4p+2}$. Because $v_{4r-3}$ and $v_{4r-1}$ in $G_r$ $(1 \leq r \leq p)$ is joined by $2p - 2$ edge-disjoint paths of length two that we call parallel paths, we can change the order of these parallel paths without changing the planarity of $G_r$. For the same reason, we can do changes like this for parallel paths between $u_{4r-1}$ and $u_{4r}$, $v_{4r-2}$ and $v_{4r}$, $u_{4r-3}$ and $u_{4r-2}$. We call this change by parallel paths modification for simplicity. All the subscripts of vertices are taken modulo $4p$, except that of $v_{4p+1}, v_{4p+2}, u_{4p+1}$ and $u_{4p+2}$ (the vertices we added to $G_r$).

**Case 1.** When $p$ is even and $p > 2$.

**(a)** The construction for $\widehat{G}_r$ , $1 \leq r \leq p$, and $r$ is odd.

**Step 1:** Place the vertex $x$ in the face 1 of $G_r$, delete edges $v_{4r-3}u_{4r}$ and $u_{4r}v_{4r-1}$ from $G_r$. Do parallel paths modification, such that $u_{4r+6} \in U_1^r$, $v_{4r+1} \in V_1^r$ and $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ are incident with a common face which the vertex $x$ is in. Join $x$ to $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ and $u_{4r+6}$, $v_{4r+1}$.

**Step 2:** Do parallel paths modification, such that $u_{4r+11}, u_{4r+12} \in U_2^r$ are incident with a common face. Place the vertex $v_{4p+1}$ in the face, and join it to both $u_{4r+11}$ and $u_{4r+12}$.

**Step 3:** Do parallel paths modification, such that $u_{4r+7}, u_{4r+8} \in U_2^r$ are incident with a common face. Place the vertex $v_{4p+2}$ in the face, and join it to both $u_{4r+7}$ and $u_{4r+8}$.

**Step 4:** Do parallel paths modification, such that $v_{4r+10}, v_{4r+12} \in V_2^r$ are incident with a common face. Place the vertex $u_{4p+1}$ in the face, and join it to both $v_{4r+10}$ and $v_{4r+12}$.

**Step 5:** Do parallel paths modification, such that $v_{4r+6}, v_{4r+8} \in V_2^r$ are incident with a common face. Place the vertex $u_{4p+2}$ in the face, and join it to both $v_{4r+6}$ and $v_{4r+8}$.

**(b)** The construction for $\widehat{G}_r$, $1 \leq r \leq p$, and $r$ is even.

**Step 1:** Place the vertex $x$ in the face 3 of $G_r$, delete edges $v_{4r}u_{4r-3}$ and $u_{4r-3}v_{4r-2}$ from $G_r$. Do parallel paths modification, such that $u_{4r+7} \in U_2^r$, $v_{4r+4} \in V_2^r$ and $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ are incident with a common face which the vertex $x$ is in. Join $x$ to $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ and $u_{4r+7}$, $v_{4r+4}$.

**Step 2:** Do parallel paths modifications, such that $u_{4r+5}, u_{4r+6} \in U_1^r$, $u_{4r+1}, u_{4r+2} \in U_1^r$, $v_{4r+5}, v_{4r+7} \in V_1^r$, $v_{4r+1}, v_{4r+3} \in V_1^r$ are incident with a common face, respectively. Join $v_{4p+1}$ to both $u_{4r+5}$ and $u_{4r+6}$, join $v_{4p+2}$ to both $u_{4r+1}$ and $u_{4r+2}$, join $u_{4p+1}$ to both $v_{4r+5}$ and $v_{4r+7}$, join $u_{4p+2}$ to both $v_{4r+1}$ and $v_{4r+3}$.

Table 1 shows how we add edges to $G_r (1 \leq r \leq p)$ in Case 1. The first column lists the edges we added, the second and third column lists the subscript of vertices, and we also indicate the vertex set which they belong to in brackets.

Table 1. The edges we add to $G_r (1 \leq r \leq p)$ in Case 1

| subscript / case / edge | $r$ is odd | | $r$ is even | |
|---|---|---|---|---|
| $xu_j$ | $4r-3, 4r-1, 4r$ | $4r+6$ $(U_1^r)$ | $4r-3, 4r-2, 4r$ | $4r+7$ $(U_2^r)$ |
| $xv_j$ | $4r-3, 4r-2, 4r-1$ | $4r+1$ $(V_1^r)$ | $4r-2, 4r-1, 4r$ | $4r+4$ $(V_2^r)$ |
| $v_{4p+1}u_j$ | $4r+11, 4r+12$ $(U_2^r)$ | | $4r+5, 4r+6$ $(U_1^r)$ | |
| $v_{4p+2}u_j$ | $4r+7, 4r+8$ $(U_2^r)$ | | $4r+1, 4r+2$ $(U_1^r)$ | |
| $u_{4p+1}v_j$ | $4r+10, 4r+12$ $(V_2^r)$ | | $4r+5, 4r+7$ $(V_1^r)$ | |
| $u_{4p+2}v_j$ | $4r+6, 4r+8$ $(V_2^r)$ | | $4r+1, 4r+3$ $(V_1^r)$ | |

**(c)** The construction for $\widehat{G}_{p+1}$.

From the construction in (a) and (b), the subscript set of $u_j$ that $xu_j$ is an edge in $\widehat{G}_r$ for some $r \in \{1, \dots, p\}$ is

$$\begin{aligned} &\{4r-3, 4r-1, 4r, 4r+6 (\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is odd}\} \\ &\cup \{4r-3, 4r-2, 4r, 4r+7 (\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is even}\} \\ &= \{1, \dots, p\}. \end{aligned}$$

The subscript set of $u_j$ that $v_{4p+1}u_j$ is an edge in $\widehat{G}_r$ for some $r \in \{1, \dots, p\}$ is

$$\begin{aligned} &\{4r+11, 4r+12 (\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is odd}\} \\ &\cup \{4r+5, 4r+6 (\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is even}\} \\ &= \{4r-3, 4r-2, 4r-1, 4r \mid 1 \leq r \leq p, \text{ and } r \text{ is even}\}. \end{aligned}$$

Using the same procedure, we can list all the edges incident with $x$, $v_{4p+1}$, $v_{4p+2}$, $u_{4p+1}$ and $u_{4p+2}$ in $\widehat{G}_r$ $(1 \leq r \leq p)$, so we can also list the edges that are incident with $x$, $v_{4p+1}$, $v_{4p+2}$, $u_{4p+1}$ in $K_{1,4p+2,4p+2}$ but not in any $\widehat{G}_r$ $(1 \leq r \leq p)$. Table 2 shows the edges that belong to $K_{1,4p+2,4p+2}$ but not to any $\widehat{G}_r$, $1 \leq r \leq p$, in which the the fourth and fifth rows list the edges deleted form $G_r$ $(1 \leq r \leq p)$ in step one of (a) and (b), and the sixth row lists the edges of $G_{p+1}$. The $\widehat{G}_{p+1}$ is the graph consists of the edges in Table 2, Figure 2 shows $\widehat{G}_{p+1}$ is a planar graph.

TABLE 2. The edges of $\widehat{G}_{p+1}$ in Case 1

| edges | subscript |
|---|---|
| $xv_{4p+1}, xu_{4p+1}, v_{4p+1}u_j, u_{4p+1}v_j$ | $j = 4r-3, 4r-2, 4r-1, 4r, 4p+2.$ $(r = 1, 3, \ldots, p-1.)$ |
| $xv_{4p+2}, xu_{4p+2}, v_{4p+2}u_j, u_{4p+2}v_j$ | $j = 4r-3, 4r-2, 4r-1, 4r, 4p+1.$ $(r = 2, 4, \ldots, p.)$ |
| $v_{4r-3}u_{4r}, u_{4r}v_{4r-1}$ | $r = 1, 3, \ldots, p-1.$ |
| $v_{4r}u_{4r-3}, u_{4r-3}v_{4r-2}$ | $r = 2, 4, \ldots, p.$ |
| $u_jv_j$ | $j = 1, \ldots, 4p+2.$ |

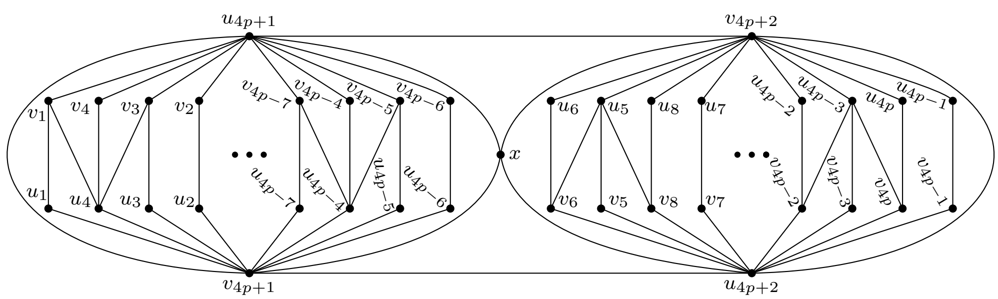
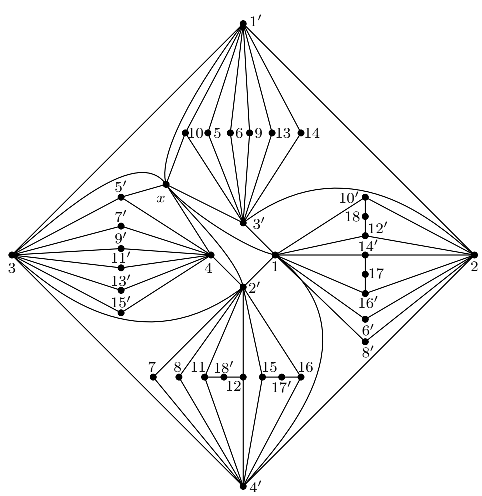
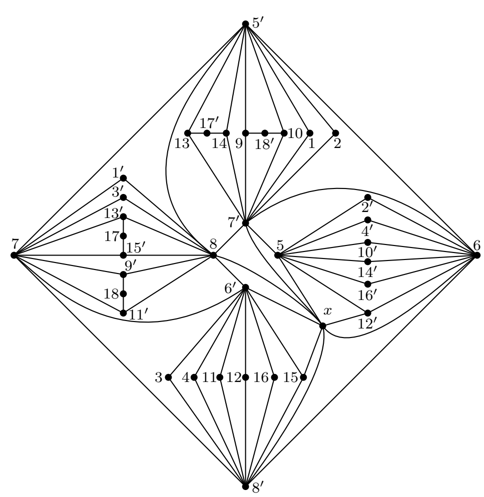



FIGURE 2. The graph $\widehat{G}_{p+1}$ in Case 1

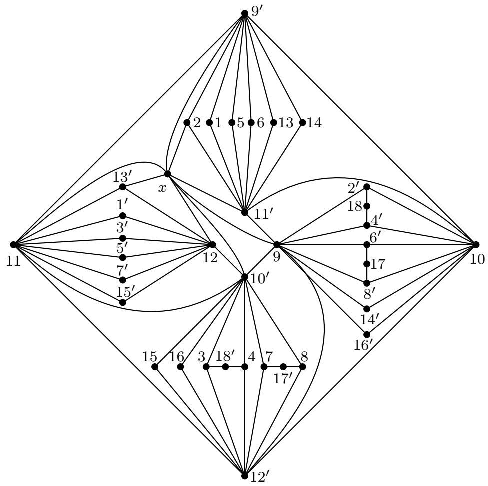
A planar decomposition $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ of $K_{1,4p+2,4p+2}$ is obtained as above in this case. In Figure 3, we draw the planar decomposition of $K_{1,18,18}$, it is the smallest example for the Case 1. We denote vertex $u_i$ and $v_i$ by $i$ and $i'$ respectively in this figure.



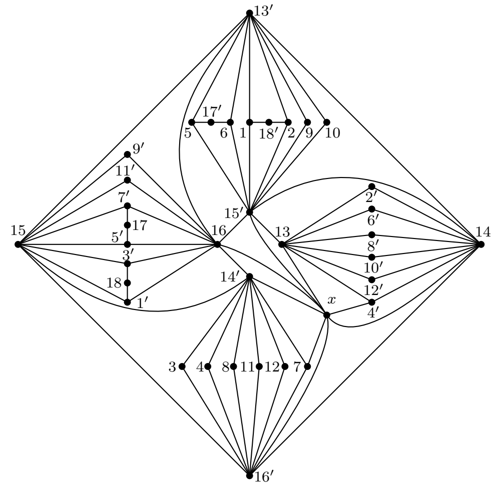

(a) The graph $\widehat{G}_1$



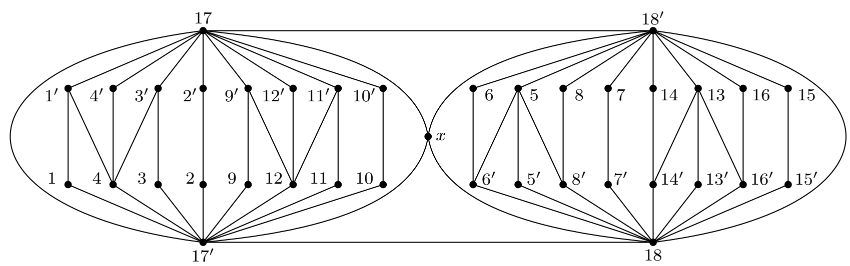

(b) The graph $\widehat{G}_2$




(c) The graph $\widehat{G}_3$




(d) The graph $\widehat{G}_4$




(e) The graph $\widehat{G}_5$

FIGURE 3. A planar decomposition of $K_{1,18,18}$

**Case 2.** When $p$ is odd and $p > 3$. The process is similar to that in Case 1.

**(a)** The construction for $\widehat{G}_r$, $1 \le r \le p$, and $r$ is odd.

**Step 1:** Place the vertex $x$ in the face 1 of $G_r$, delete edges $v_{4r-3}u_{4r}$ and $u_{4r}v_{4r-1}$ from $G_r$, for $1 \le r \le p$, and delete $v_2u_1$ from $G_1$ additionally.

For $1 < r < p$, do parallel paths modification to $G_r$, such that $u_{4r+6} \in U_1^r$, $v_{4r+1} \in V_1^r$ and $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ are incident with a common face which the vertex $x$ is in. Join $x$ to $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ and $u_{4r+6}$, $v_{4r+1}$.

Similarly, in $G_1$, join $x$ to $u_1, u_3, u_4, v_1, v_2, v_3, v_4$ and $u_{10} \in U_1^1$, $v_5 \in V_1^1$. In $G_p$, join $x$ to $u_{4p-3}$,$u_{4p-1}$,$u_{4p}$,$v_{4p-3}$,$v_{4p-2}$,$v_{4p-1}$ and $u_2 \in U_1^p$.

**Step 2:** For $1 \le r < p$, do parallel paths modification to $G_r$, such that $u_{4r+11}$, $u_{4r+12} \in U_2^r$, $u_{4r+7}, u_{4r+8} \in U_2^r$, $v_{4r+10}, v_{4r+12} \in V_2^r$ and $v_{4r+6}, v_{4r+8} \in V_2^r$ are incident with a common face, respectively. Join $v_{4p+1}$ to both $u_{4r+11}$ and $u_{4r+12}$, join $v_{4p+2}$ to both $u_{4r+7}$ and $u_{4r+8}$, join $u_{4p+1}$ to both $v_{4r+10}$ and $v_{4r+12}$, join $u_{4p+2}$ to both $v_{4r+6}$ and $v_{4r+8}$.

Similarly, in $G_p$, join $v_{4p+1}$ to $u_5, u_6 \in U_1^p$, join $v_{4p+2}$ to $u_7, u_8 \in U_2^p$, join $u_{4p+1}$ to $v_6, v_8 \in V_2^p$, join $u_{4p+2}$ to $v_5, v_7 \in V_1^p$.

**(b)** The construction for $\widehat{G}_r$, $1 \le r \le p$, and $r$ is even.

**Step 1:** Place the vertex $x$ in the face 3 of $G_r$, delete edges $v_{4r}u_{4r-3}$ and $u_{4r-3}v_{4r-2}$ from $G_r$, $1 \le r \le p-1$.

Do parallel paths modification to $G_r, 1 \le r < p-1$, such that $u_{4r+7} \in U_2^r$, $v_{4r+4} \in V_2^r$ and $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ are incident with a common face which the vertex $x$ is in. Join $x$ to $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ and $u_{4r+7}$, $v_{4r+4}$. Similarly, in $G_{p-1}$, join $x$ to $u_{4p-7}, u_{4p-6}, u_{4p-4}, v_{4p-6}, v_{4p-5}, v_{4p-4}$ and $u_7 \in U_2^{p-1}$, $v_{4p} \in V_2^{p-1}$.

**Step 2:** Do parallel paths modifications, such that $u_{4r+5}, u_{4r+6} \in U_1^r$, $u_{4r+1}, u_{4r+2} \in U_1^r$, $v_{4r+5}, v_{4r+7} \in V_1^r$, $v_{4r+1}, v_{4r+3} \in V_1^r$ are incident with a common face, respectively. Join $v_{4p+1}$ to both $u_{4r+5}$ and $u_{4r+6}$, join $v_{4p+2}$ to both $u_{4r+1}$ and $u_{4r+2}$, join $u_{4p+1}$ to both $v_{4r+5}$ and $v_{4r+7}$, join $u_{4p+2}$ to both $v_{4r+1}$ and $v_{4r+3}$.

Table 3 shows how we add edges to $G_r(1 \le r \le p)$ in Case 2.

TABLE 3. The edges we add to $G_r(1 \le r \le p)$ in Case 2

| subscript \ case / edge | $r$ is odd | | $r$ is even | |
|---|---|---|---|---|
| $xu_j$ | $4r-3, 4r-1, 4r$ | $4r+6, r \neq p$ $(U_1^r)$<br>$2, r = p$ $(U_1^r)$ | $4r-3, 4r-2, 4r$ | $4r+7, r \neq p-1$ $(U_2^r)$<br>$7, r = p-1$ $(U_2^r)$ |
| $xv_j$ | $4r-3, 4r-2, 4r-1$ | $4, 5, r = 1$<br>$4r+1, r \neq 1, p$ $(V_1^r)$ | $4r-2, 4r-1, 4r$ | $4r+4$ $(V_2^r)$ |
| $v_{4p+1}u_j$ | $4r+11, 4r+12, r \neq p$ $(U_2^r)$<br>$5, 6, r = p$ $(U_1^r)$ | | $4r+5, 4r+6$ $(U_1^r)$ | |
| $v_{4p+2}u_j$ | $4r+7, 4r+8$ $(U_2^r)$ | | $4r+1, 4r+2$ $(U_1^r)$ | |
| $u_{4p+1}v_j$ | $4r+10, 4r+12, r \neq p$ $(V_2^r)$<br>$6, 8, r = p$ $(V_2^r)$ | | $4r+5, 4r+7$ $(V_1^r)$ | |
| $u_{4p+2}v_j$ | $4r+6, 4r+8, r \neq p$ $(V_2^r)$<br>$5, 7, r = p$ $(V_1^r)$ | | $4r+1, 4r+3$ $(V_1^r)$ | |

**(c)** The construction for $\widehat{G}_{p+1}$.

With a similar argument to that in Case 1, we can list the edges that belong to $K_{1,4p+2,4p+2}$ but not to any $\widehat{G}_r$, $1 \le r \le p$, in this case, as shown in Table 4. Then $\widehat{G}_{p+1}$ is the graph that consists of the edges in Table 4, Figure 4 shows $\widehat{G}_{p+1}$ is a planar graph.

Therefore, $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{1,4p+2,4p+2}$ in this case.

**Case 3.** When $p \le 3$.

When $p = 0$, $K_{1,2,2}$ is a planar graph. When $p = 1, 2, 3$, we give a planar decomposition for $K_{1,6,6}$, $K_{1,10,10}$ and $K_{1,14,14}$ with 2, 3 and 4 subgraphs respectively, as shown in Figure 5, Figure 6 and Figure 7.

TABLE 4. The edges of $\widehat{G}_{p+1}$ in Case 2

| edges | subscript |
|---|---|
| $xv_{4p+1}, v_{4p+1}u_j$ | $j = 4r-3, 4r-2, 4r-1, 4r, 7, 8, 4p+2$. $(r = 3, 5, 7, \ldots, p.)$ |
| $xu_{4p+1}, u_{4p+1}v_j$ | $j = 4r-3, 4r-2, 4r-1, 4r, 5, 7, 4p+2$. $(r = 3, 5, 7, \ldots, p.)$ |
| $xv_{4p+2}, v_{4p+2}u_j$ | $j = 4r-3, 4r-2, 4r-1, 4r, 5, 6, 4p+1$. $(r = 1, 4, 6, 8 \ldots, p-1.)$ |
| $xu_{4p+2}, u_{4p+2}v_j$ | $j = 4r-3, 4r-2, 4r-1, 4r, 6, 8, 4p+1$. $(r = 1, 4, 6, 8 \ldots, p-1.)$ |
| $u_1v_2, v_{4r-3}u_{4r}, u_{4r}v_{4r-1}$ | $r = 1, 3, \ldots, p$. |
| $v_{4r}u_{4r-3}, u_{4r-3}v_{4r-2}$ | $r = 2, 4, \ldots, p-1$. |
| $u_jv_j$ | $j = 1, \ldots, 4p+2$. |

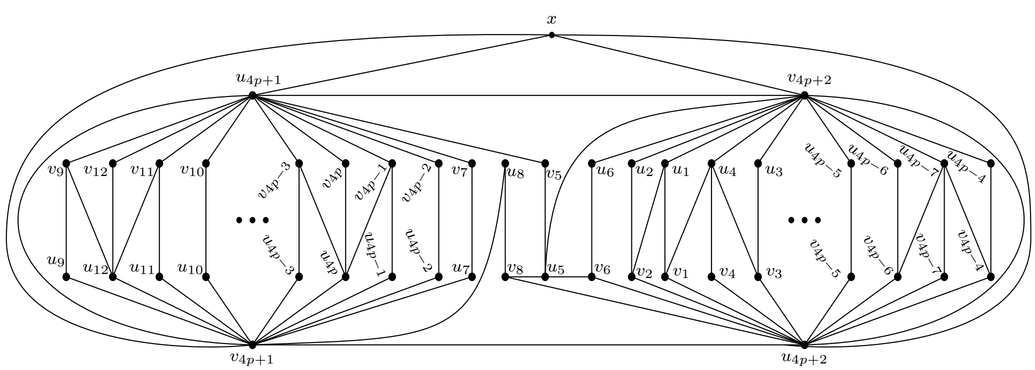



FIGURE 4. The graph $\widehat{G}_{p+1}$ in Case 2

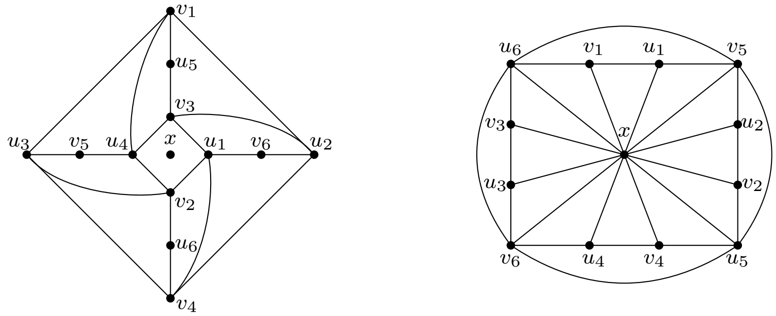



FIGURE 5. A planar decomposition of $K_{1,6,6}$

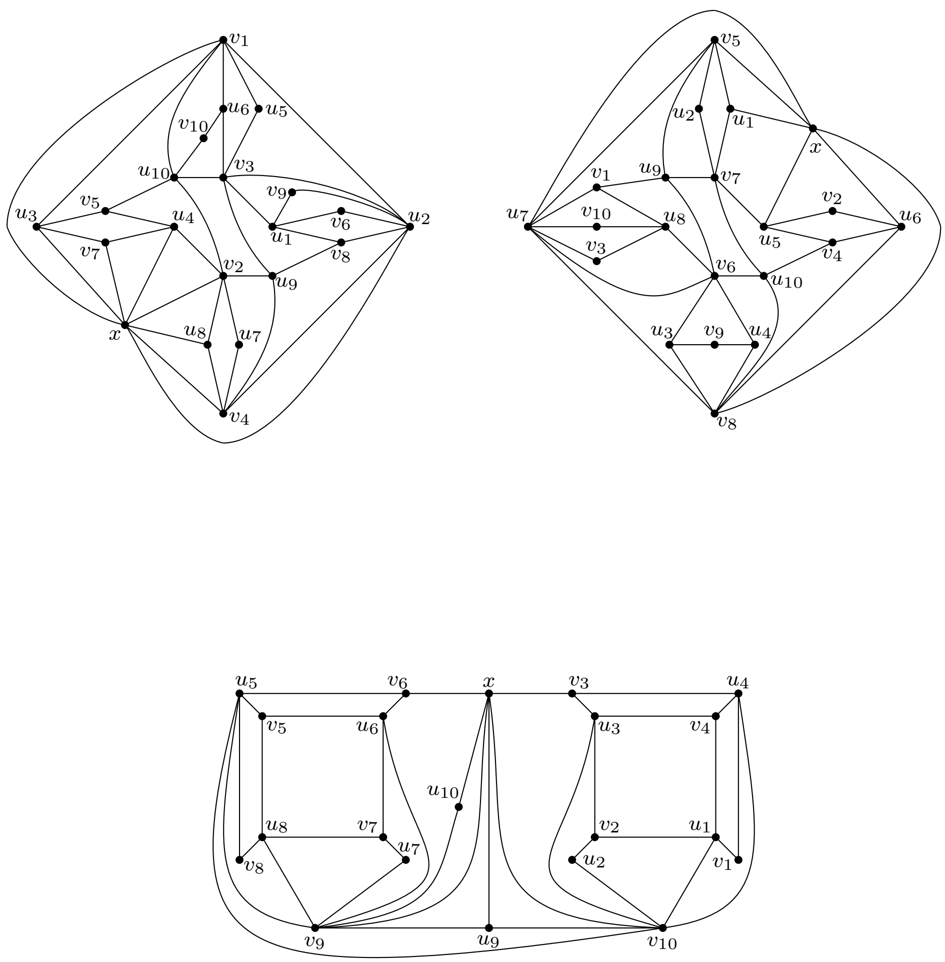
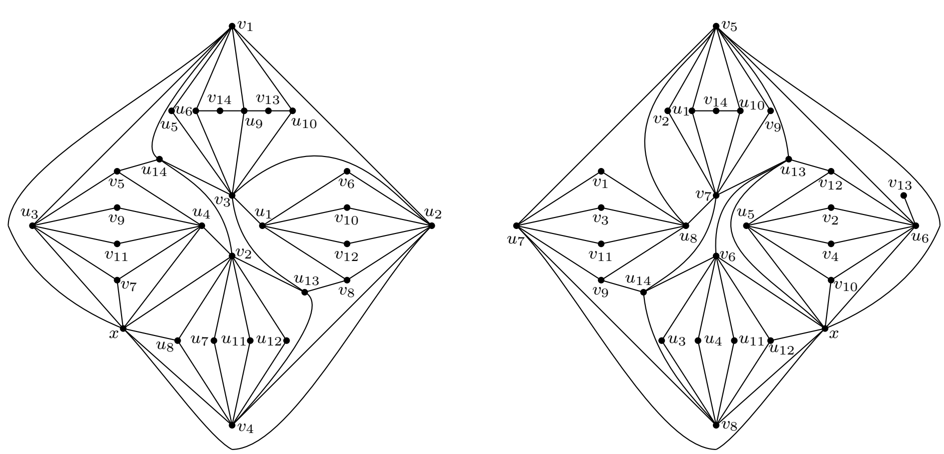



FIGURE 6. A planar decomposition of $K_{1,10,10}$



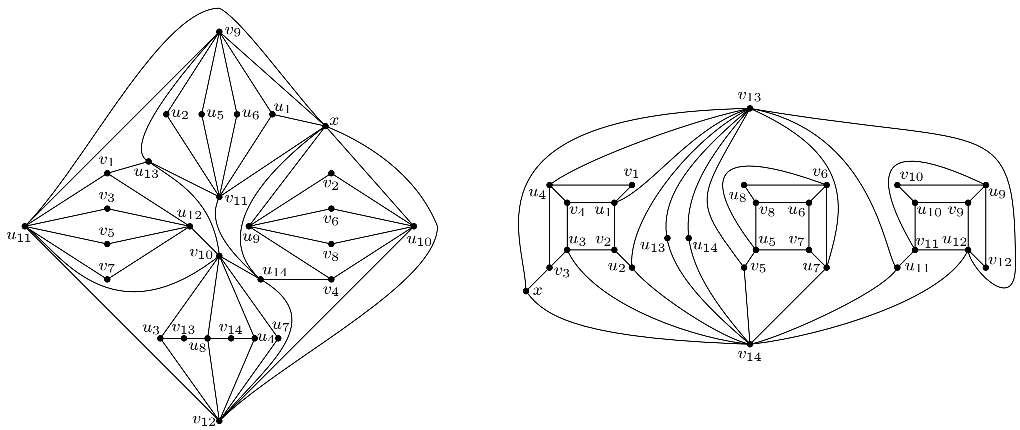



FIGURE 7. A planar decomposition of $K_{1,14,14}$

Lemma follows from Cases 1, 2 and 3. □

**Theorem 2.4.** *The thickness of the complete 3-partite graph $K_{1,n,n}$ is*

$$\theta(K_{1,n,n}) = \lceil \frac{n+2}{4} \rceil.$$

*Proof.* When $n = 4p, 4p+3$, the theorem follows from Lemma 2.2.

When $n = 4p+1, n = 4p+2$, from Lemma 2.3, we have $\theta(K_{1,4p+2,4p+2}) \leq p+1$. Since $\theta(K_{4p,4p}) = p+1$ and $K_{4p,4p} \subset K_{1,4p+1,4p+1} \subset K_{1,4p+2,4p+2}$, we obtain

$$p+1 \leq \theta(K_{1,4p+1,4p+1}) \leq \theta(K_{1,4p+2,4p+2}) \leq p+1.$$

Therefore, $\theta(K_{1,4p+1,4p+1}) = \theta(K_{1,4p+2,4p+2}) = p+1$.

Summarizing the above, the theorem is obtained. □

## 3. THE THICKNESS OF $K_{2,n,n}$

**Lemma 3.1.** *There exists a planar decomposition of the complete 3-partite graph $K_{2,4p+1,4p+1}$ $(p \geq 0)$ with $p+1$ subgraphs.*

*Proof.* Let $(X, U, V)$ be the vertex partition of the complete 3-partite graph $K_{2,n,n}$, in which $X = \{x_1, x_2\}$, $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$. When $n = 4p+1$, we will construct a planar decomposition of $K_{2,4p+1,4p+1}$ with $p+1$ planar subgraphs.

The construction is analogous to that in Lemma 2.3. Let $\{G_1, G_2, \ldots, G_{p+1}\}$ be a planar decomposition of $K_{4p,4p}$ given in [4]. In the following, for $1 \leq r \leq p+1$, by adding vertices $x_1, x_2, u_{4p+1}, v_{4p+1}$ to $G_r$, deleting some edges from $G_r$ and adding some edges to $G_r$, we will get a new planar graph $\widehat{G}_r$ such that $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$. All the subscripts of vertices are taken modulo $4p$, except that of $u_{4p+1}$ and $v_{4p+1}$ (the vertices we added to $G_r$).

**Case 1.** When $p$ is even and $p > 2$.

**(a)** The construction for $\widehat{G}_r$ , $1 \leq r \leq p$.

**Step 1:** When $r$ is odd, place the vertex $x_1$,$x_2$ and $u_{4p+1}$ in the face 1,2 and 5 of $G_r$ respectively. Delete edges $v_{4r-3}u_{4r}$ and $u_{4r-1}v_{4r-2}$ from $G_r$.

When $r$ is even, place the vertex $x_1$,$x_2$ and $u_{4p+1}$ in the face 3,4 and 5 of $G_r$, respectively. Delete edge $v_{4r}u_{4r-3}$ and $u_{4r-2}v_{4r-1}$ from $G_r$.

**Step 2:** Do parallel paths modifications, then join $x_1$, $x_2$, $u_{4p+1}$ and $v_{4p+1}$ to some $u_j$ and $v_j$, as shown in Table 5.

TABLE 5. The edges we add to $G_r(1 \le r \le p)$ in Case 1

| subscript \ case / edge | $r$ is odd | | $r$ is even | |
|---|---|---|---|---|
| $x_1u_j$ | $4r-1, 4r$ | $4r+5$ ($U_1^r$) | $4r-3, 4r-2$ | $4r+8$ ($U_2^r$) |
| $x_1v_j$ | $4r-3, 4r-1$ | $4r+1$ ($V_1^r$) | $4r-2, 4r$ | $4r+4$ ($V_2^r$) |
| $x_2u_j$ | $4r-1, 4r$ | $4r+3$ ($U_2^r$) | $4r-3, 4r-2$ | $4r+2$ ($U_1^r$) |
| $x_2v_j$ | $4r-2, 4r$ | $4r+7$ ($V_1^r$) | $4r-3, 4r-1$ | $4r+6$ ($V_2^r$) |
| $u_{4p+1}v_j$ | $4r-2, 4r-1$ | | | |
| $v_{4p+1}u_j$ | $4r+4, 4r+8$ ($U_2^r$) | | $4r-11, 4r-7$ ($U_1^r$) | |

**(b)** The construction for $\widehat{G}_{p+1}$.

We list the edges that belong to $K_{2,4p+1,4p+1}$ but not to any $\widehat{G}_r$, $1 \le r \le p$, as shown in Table 6. Then $\widehat{G}_{p+1}$ is the graph that consists of the edges in Table 6, Figure 8 shows $\widehat{G}_{p+1}$ is a planar graph.

TABLE 6. The edges of $\widehat{G}_{p+1}$ in Case 1

| edges | subscript |
|---|---|
| $x_1u_j$, $x_1v_j$ | $j = 4r-2, 4r+3, 4p+1$. ($r = 1, 3, \dots, p-1$.) |
| $x_2u_j$, $x_2v_j$ | $j = 4r-7, 4r, 4p+1$. ($r = 2, 4, \dots, p$.) |
| $u_{4p+1}v_j$ | $j = 4r-3, 4r$. ($r = 1, 2, \dots, p$.) |
| $v_{4p+1}u_j$ | $j = 4r-2, 4r-1$. ($r = 1, 2, \dots, p$.) |
| $v_{4r-3}u_{4r}, v_{4r-2}u_{4r-1}$ | $r = 1, 3, \dots, p-1$. |
| $u_{4r-3}v_{4r}, u_{4r-2}v_{4r-1}$ | $r = 2, 4, \dots, p$. |
| $u_jv_j$ | $j = 1, \dots, 4p+1$. |

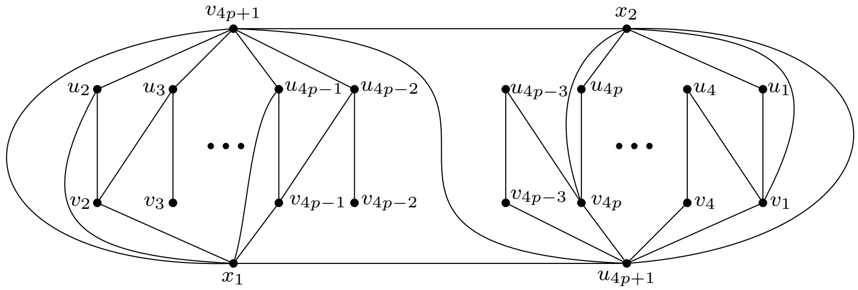
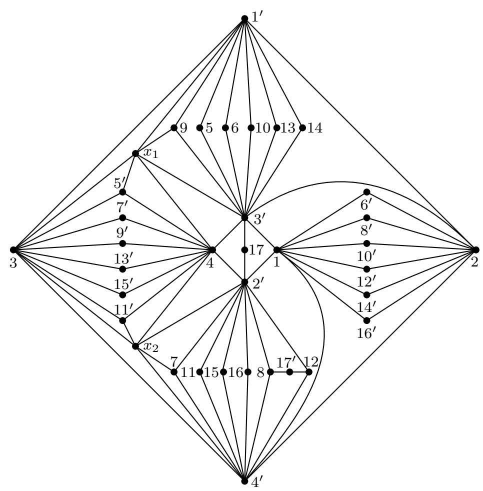
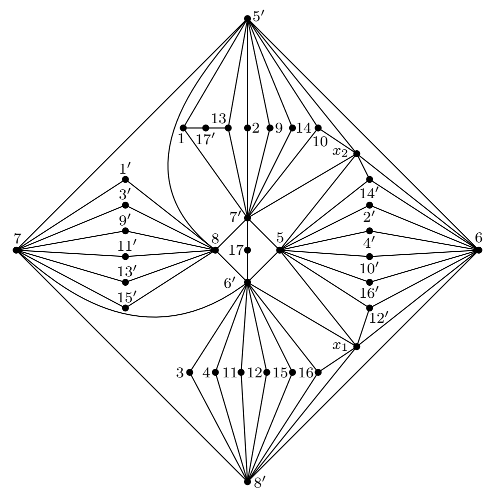



FIGURE 8. The graph $\widehat{G}_{p+1}$ in Case 1

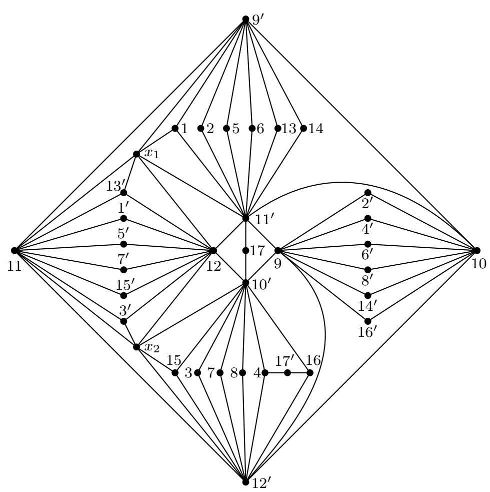
Therefore, $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$ in this case. In Figure 9, we draw the planar decomposition of $K_{2,17,17}$ it is the smallest example for the Case 1. We denote vertex $u_i$ and $v_i$ by $i$ and $i'$ respectively in this figure.



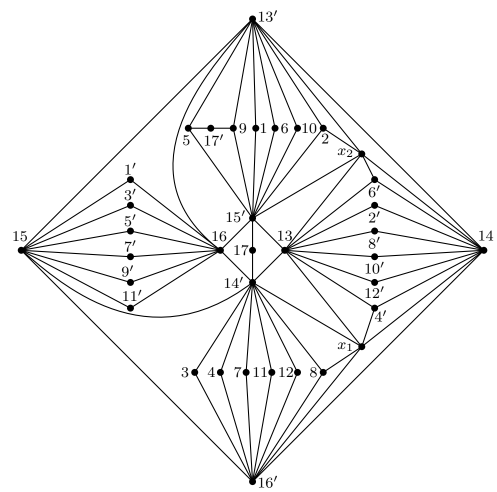
(a) The graph $\widehat{G}_1$



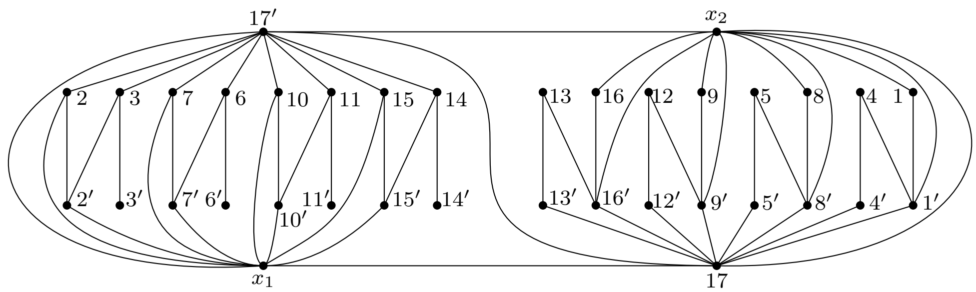
(b) The graph $\widehat{G}_2$



(c) The graph $\widehat{G}_3$



(d) The graph $\widehat{G}_4$




(e) The graph $\widehat{G}_5$

FIGURE 9. A planar decomposition of $K_{2,17,17}$

**Case 2.** When $p$ is odd and $p > 3$.

**(a)** The construction for $\widehat{G}_r$ , $1 \le r \le p$.

**Step 1:** When $r$ is odd, place the vertex $x_1$,$x_2$ and $u_{4p+1}$ in the face 1,2 and 5 of $G_r$ respectively. Delete edges $v_{4r-3}u_{4r}$ and $u_{4r-1}v_{4r-2}$ from $G_r$.

When $r$ is even, place the vertex $x_1$,$x_2$ and $u_{4p+1}$ in the face 3,4 and 5 of $G_r$, respectively. Delete edge $v_{4r}u_{4r-3}$ and $u_{4r-2}v_{4r-1}$ from $G_r$.

**Step 2:** Do parallel paths modifications, then join $x_1$, $x_2$, $u_{4p+1}$ and $v_{4p+1}$ to some $u_j$ and $v_j$, as shown in Table 7.

TABLE 7. The edges we add to $G_r (1 \le r \le p)$ in Case 2

| subscript / case / edge | $r$ is odd | | $r$ is even | |
|---|---|---|---|---|
| $x_1u_j$ | $4r-1, 4r$ | $4r+5, r \ne p$ $(U_1^r)$<br>$1, r = p$ $(U_1^r)$ | $4r-3, 4r-2$ | $4r+8, r \ne p-1$ $(U_2^r)$<br>$8, r = p-1$ $(U_2^r)$ |
| $x_1v_j$ | $4r-3, 4r-1$ | $4r+1, r \ne p$ $(V_1^r)$ | $4r-2, 4r$ | $4r+4$ $(V_2^r)$ |
| $x_2u_j$ | $4r-1, 4r$ | $4r+3, r \ne p$ $(U_2^r)$<br>$8, r = p$ $(U_2^r)$ | $4r-3, 4r-2$ | $4r+2$ $(U_1^r)$ |
| $x_2v_j$ | $4r-2, 4r$ | $4r+7, r \ne p$ $(V_1^r)$<br>$3, r = p$ $(V_1^r)$ | $4r-3, 4r-1$ | $4r+6, r \ne p-1$ $(V_2^r)$<br>$6, r = p-1$ $(V_2^r)$ |
| $u_{4p+1}v_j$ | $4r-2, 4r-1$ | | | |
| $v_{4p+1}u_j$ | $4r+4, 4r+8, r \ne p$ $(U_2^r)$<br>$4, r = p$ $(U_2^r)$ | | $4r-11, 4r-7$ $(U_1^r)$ | |

**(b)** The construction for $\widehat{G}_{p+1}$.

We list the edges that belong to $K_{2,4p+1,4p+1}$ but not to any $\widehat{G}_r$, $1 \le r \le p$, as shown in Table 8. Then $\widehat{G}_{p+1}$ is the graph that consists of the edges in Table 8, Figure 10 shows $\widehat{G}_{p+1}$ is a planar graph.

TABLE 8. The edges of $\widehat{G}_{p+1}$ in Case 2

| edges | subscript |
|---|---|
| $x_1u_j$ | $j = 2, 4r+3, 4r+6, 4p+1$. $(r = 1, 3, \ldots, p-2.)$ |
| $x_1v_j$ | $j = 2, 4, 4r+3, 4r+6, 4p+1$. $(r = 1, 3, \ldots, p-2.)$ |
| $x_2u_j$ | $j = 1, 2, 9, 4r, 4r+1, 4p+1$. $(r = 4, \ldots, p-1.)$ |
| $x_2v_j$ | $j = 1, 8, 9, 4r, 4r+1, 4p+1$. $(r = 4, \ldots, p-1.)$ |
| $u_{4p+1}v_j$ | $j = 4r-3, 4r.(r = 1, 2, \ldots, p.)$ |
| $v_{4p+1}u_j$ | $j = 4r-2, 4r-1, 4p-7.(r = 1, 2, \ldots, p.)$ |
| $v_{4r-3}u_{4r}, v_{4r-2}u_{4r-1}$ | $r = 1, 3, \ldots, p.$ |
| $u_{4r-3}v_{4r}, u_{4r-2}v_{4r-1}$ | $r = 2, 4, \ldots, p-1.$ |
| $u_jv_j$ | $j = 1, \ldots, 4p+1.$ |

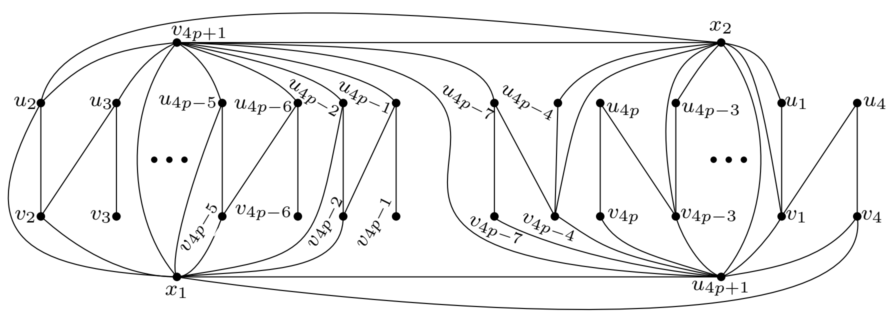



FIGURE 10. The graph $\widehat{G}_{p+1}$ in Case 2

Therefore, $\{\widehat{G}_1, \ldots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$ in this case.

**Case 3.** When $p \leq 3$.

When $p = 0$, $K_{2,1,1}$ is a planar graph. When $p = 1, 2, 3$, we give a planar decomposition for $K_{2,5,5}$, $K_{2,9,9}$ and $K_{2,13,13}$ with 2, 3 and 4 subgraphs respectively, as shown in Figure 11, Figure 12 and Figure 13.

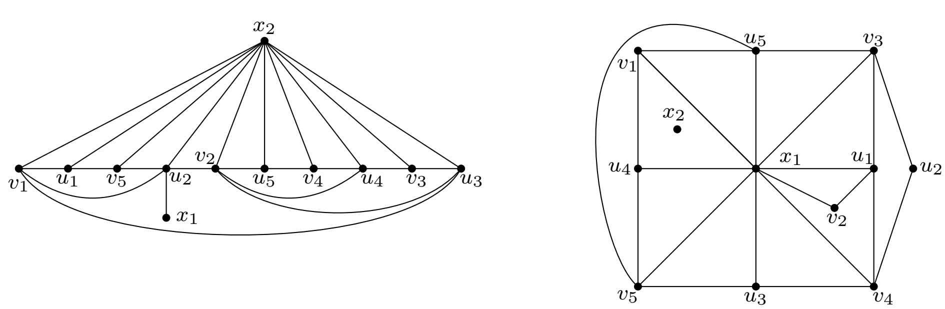



FIGURE 11. A planar decomposition $K_{2,5,5}$

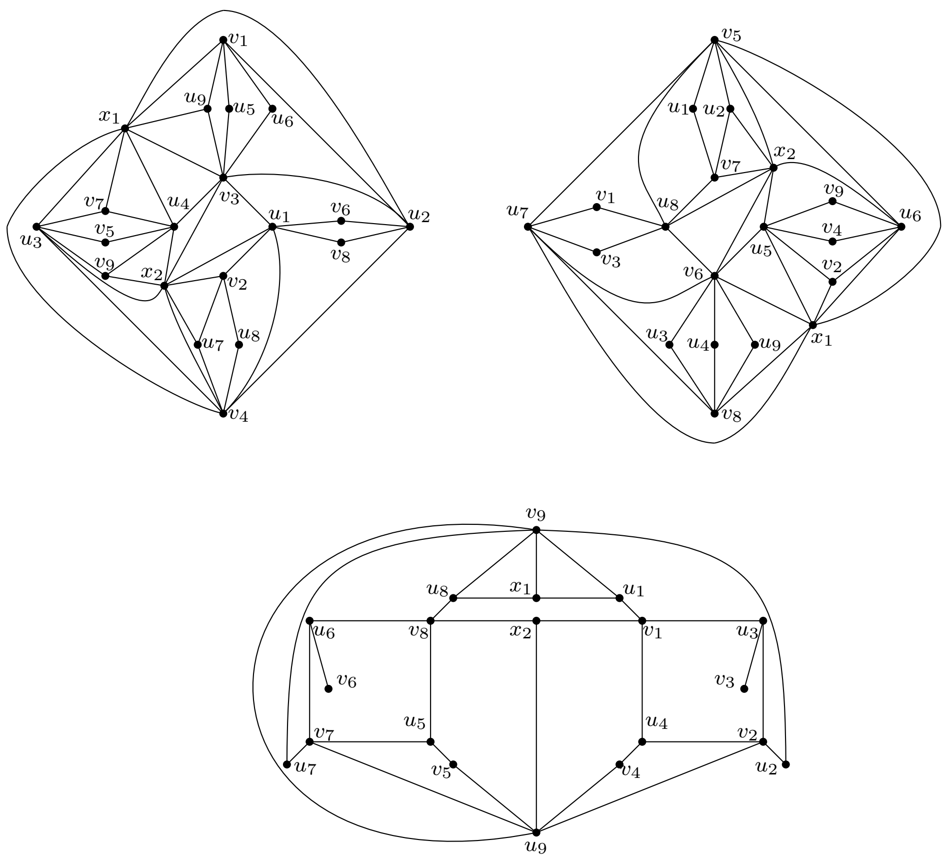
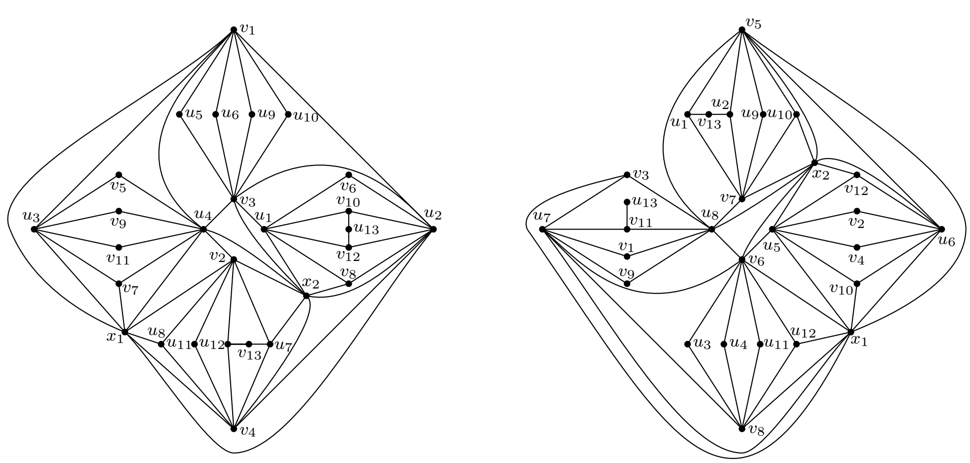



FIGURE 12. A planar decomposition $K_{2,9,9}$



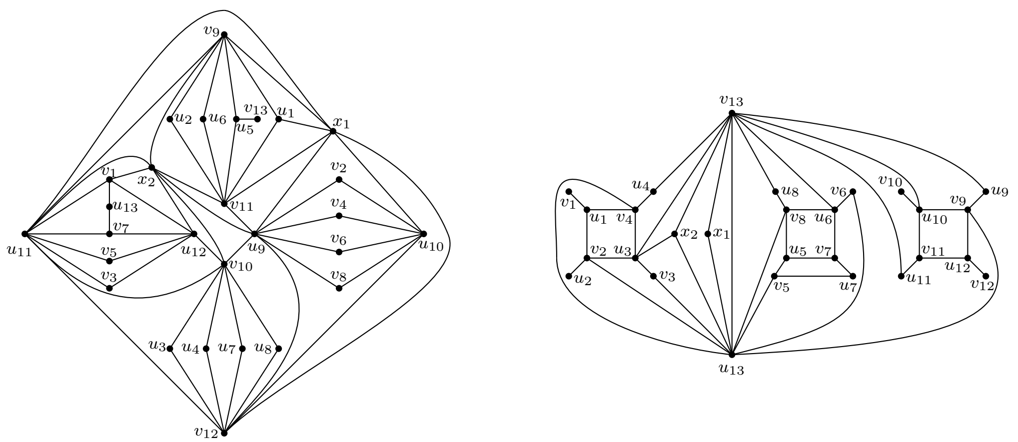



FIGURE 13. A planar decomposition of $K_{2,13,13}$

Summarizing Cases 1,2 and 3, the lemma follows. □

**Theorem 3.2.** *The thickness of the complete 3-partite graph $K_{2,n,n}$ is*

$$\theta(K_{2,n,n}) = \lceil \frac{n+3}{4} \rceil.$$

*Proof.* When $n = 4p, 4p+3$, from Lemma 2.2, the theorem holds.

When $n = 4p+1$, from Lemma 3.1, we have $\theta(K_{2,4p+1,4p+1}) \leq p+1$. Since $\theta(K_{4p,4p}) = p+1$ and $K_{4p,4p} \subset K_{2,4p+1,4p+1}$, we have

$$p+1 = \theta(K_{4p,4p}) \leq \theta(K_{2,4p+1,4p+1}) \leq p+1.$$

Therefore, $\theta(K_{2,4p+1,4p+1}) = p+1$.

When $n = 4p+2$, since $K_{4p+3,4p+3} \subset K_{2,4p+2,4p+2}$, from Lemma 2.1, we have $p+2 = \theta(K_{4p+3,4p+3}) \leq \theta(K_{2,4p+2,4p+2})$. On the other hand, it is easy to see $\theta(K_{2,4p+2,4p+2}) \leq \theta(K_{2,4p+1,4p+1}) + 1 = p+2$, so we have $\theta(K_{2,4p+2,4p+2}) = p+2$.

Summarizing the above, the theorem is obtained. □

## 4. The thickness of $K_{1,1,n,n}$

**Theorem 4.1.** *The thickness of the complete 4-partite graph $K_{1,1,n,n}$ is*

$$\theta(K_{1,1,n,n}) = \lceil \frac{n+3}{4} \rceil.$$

*Proof.* When $n = 4p+1$, we can get a planar decomposition for $K_{1,1,4p+1,4p+1}$ from that of $K_{2,4p+1,4p+1}$ as follows.

(1) When $p = 0$, $K_{1,1,1,1}$ is a planar graph, $\theta(K_{1,1,1,1}) = 1$. When $p = 1, 2$ and 3, we join the vertex $x_1$ to $x_2$ in the last planar subgraph in the planar decomposition for $K_{2,5,5}$, $K_{2,9,9}$ and $K_{2,13,13}$ which was shown in Figure 11, 12 and 13. Then we get the planar decomposition for $K_{1,1,5,5}$, $K_{1,1,9,9}$ and $K_{1,1,13,13}$ with 2, 3 and 4 planar subgraphs respectively.

(2) When $p \geq 4$, we join the vertex $x_1$ to $x_2$ in $\widehat{G}_{p+1}$ in the planar decomposition for $K_{2,4p+1,4p+1}$ which was constructed in Lemma 3.1. The $\widehat{G}_{p+1}$ is shown

in Figure 8 or 10 according to $p$ is even or odd. Because $x_1$ and $x_2$ lie on the boundary of the same face, we will get a planar graph by adding edge $x_1x_2$ to $\widehat{G}_{p+1}$. Then a planar decomposition for $K_{1,1,4p+1,4p+1}$ with $p+1$ planar subgraphs can be obtained.

Summarizing (1) and (2), we have $K_{1,1,4p+1,4p+1} \leq p+1$.

On the other hand, from Lemma 2.1, we have $\theta(K_{4p+1,4p+1}) = p+1$. Due to $K_{4p+1,4p+1} \subset K_{1,1,4p,4p} \subset K_{1,1,4p+1,4p+1}$, we get $p+1 \leq \theta(K_{1,1,4p,4p}) \leq \theta(K_{1,1,4p+1,4p+1})$. So we have

$$\theta(K_{1,1,4p,4p}) = \theta(K_{1,1,4p+1,4p+1}) = p+1.$$

When $n = 4p+3$, from Theorem 3.2 , we have $\theta(K_{2,4p+2,4p+2}) = p+2$. Since $K_{2,4p+2,4p+2} \subset K_{1,1,4p+2,4p+2} \subset K_{1,1,4p+3,4p+3} \subset K_{1,1,4(p+1),4(p+1)}$, and the ideas from the previous case establish, we have $p+2 \leq \theta(K_{1,1,4p+2,4p+2}) \leq \theta(K_{1,1,4p+3,4p+3}) \leq \theta(K_{1,1,4(p+1),4(p+1)}) = p+2$, which shows

$$\theta(K_{1,1,4p+2,4p+2}) = \theta(K_{1,1,4p+3,4p+3}) = p+2.$$

Summarizing the above, the theorem follows. □

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Department of Mathematics, Tianjin University, 300072, Tianjin, China

*E-mail address*: guoxia@tju.edu.cn

Department of Mathematics, Tianjin University, 300072, Tianjin, China

*E-mail address*: yanyang@tju.edu.cn (Corresponding author: Yan YANG)