AN INVERSE PROBLEM FOR NON-SELFADJOINT STURM-LIOUVILLE OPERATOR WITH DISCONTINUITY CONDITIONS INSIDE A FINITE INTERVAL

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Abstract. This paper is concerned with the inverse problem for non-selfadjoint Sturm-Liouville operator with discontinuity conditions inside a finite interval. Firstly, we give the definitions of generalized weight numbers for this operator which may have the multiple spectrum, and then investigate the connections between the generalized weight numbers and other spectral characteristics. Secondly, we obtain the generalized spectral data, which consists of the generalized weight numbers and the spectrum. Then the operator is determined uniquely by the method of spectral mappings. Finally, we give an algorithm for reconstructing the potential function and the coefficients of the boundary conditions and the coefficients of the discontinuity conditions.

1. Introduction

In this paper, we consider the following non-selfadjoint boundary value problem $L = L\left(q\left(x\right), h, H, \beta, \gamma, d\right)$ for the equation:

$$\ell y := -y'' + q\left(x\right) y = \lambda y \tag{1.1}$$

on the interval $0 < x < \pi$ with the boundary conditions

$$U\left(y\right) := y'\left(0\right) - hy\left(0\right) = 0, V\left(y\right) := y'\left(\pi\right) + Hy\left(\pi\right) = 0 \tag{1.2}$$

and the discontinuity conditions

$$y\left(d+0\right) = \beta y\left(d-0\right), \; y'\left(d+0\right) = \beta^{-1} y'\left(d-0\right) + \gamma y\left(d-0\right) \tag{1.3}$$

at $d \in (0, \pi)$, where $q(x) \in L^2[0, \pi]$ is a complex-valued function, h, H, γ are complex numbers, and $\beta \in \mathbb{R}$, $\beta \neq 0$.

There has been extensive study of inverse problems for Sturm-Liouville operator with discontinuity conditions inside a finite interval since the discontinuities are connected with non-smooth material properties. The inverse problem for selfadjoint Sturm-Liouville operator with different type discontinuity has been considered and solved by different methods in [8, 9, 16, 18, 19, 22–26]. [17, 21] studied the inverse spectral problem for discontinuous Sturm-Liouville operators with boundary conditions linearly dependent on the spectral parameter. The inverse problem for non-selfadjoint Sturm-Liouville operator with discontinuity inside an interval has been investigated in [12, 15] when the spectrum is simple.

Recently, many authors paid more attention on the inverse problem for the non-selfadjoint operator with multiple spectrum (see [1, 4, 5, 10, 14, 20] and the references therein). Especially, Buterin [5] considered the inverse problem for the

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boundary value problem (1.1), (1.2) with an arbitrary behaviour of the spectrum and gave generalized weight numbers more naturally and proved that a multiple spectrum and the generalized weight numbers determine the potential function and boundary conditions uniquely. In this paper, we add the discontinuity conditions (1.3) at $d \in (0, \pi)$ to the boundary value problem (1.1), (1.2), and give the generalized weight numbers for discontinuous non-selfadjoint Sturm-Liouville operator with multiple spectrum, and recovering this operator from its spectral characteristics by spectral mappings (see [27]).

This paper is organized as follows. In Section 2, some basic definitions and useful properties are given. We devote Section 3 to give the useful definition of generalized spectral data. The connections between the generalized spectral data and other spectral characteristics are investigated in Section 4. In Section 5, by the method of spectral mappings, we prove that the given generalized spectral data uniquely determine the potential q and the coefficients h, H, β, γ, respectively, and then give an algorithm for reconstructing the operator $L(q(x), h, H, \beta, \gamma, d)$.

2. Preliminaries

Let $y(x)$, $z(x)$ be continuously differentiable functions on $[0, d]$ and $[d, \pi]$. Denote $\langle y(x), z(x)\rangle := y(x) z'(x) - y'(x) z(x)$. If $y(x)$ and $z(x)$ satisfy the discontinuity conditions (1.3), then

$$\langle y(x), z(x)\rangle_{x=d-0} = \langle y(x), z(x)\rangle_{x=d+0}, \tag{2.1}$$

Let $\varphi(x, \lambda)$, $\psi(x, \lambda)$ be solutions of equation (1.1) satisfying the discontinuity conditions (1.3) and the initial conditions

$$\varphi(0, \lambda) = \psi(\pi, \lambda) = 1, \varphi'(0, \lambda) = h, \psi'(\pi, \lambda) = -H, \tag{2.2}$$

respectively. Then $U(\varphi) = V(\psi) = 0$. Denote $\Delta(\lambda) := \langle \varphi(x, \lambda), \psi(x, \lambda)\rangle$, then $\Delta(\lambda)$ is independent of x. From (2.2), we obtain

$$\Delta(\lambda) = -V(\varphi) = U(\psi). \tag{2.3}$$

In the following, we give three powerful and important lemmas, the rigourous proof of these lemmas which can be referred to [2, 3, 6, 11, 13, 25] and no proof will be given here.

Lemma 1. *The zeros of $\Delta(\lambda)$ coincide with the eigenvalues λ_n, $n \in \mathbb{N} := \{0, 1, 2, ..., n, ...\}$ of L. $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are corresponding eigenfunctions of L.*

Proof. See [5, p.740] and [12, p.3]. □

Lemma 2. *Let $\rho = \sqrt{\lambda}$, $\tau = \mathrm{Im}\rho$.* For $|\rho| \to \infty$,

$$\varphi(x, \lambda) = \begin{cases} \cos\rho x + O\left(\frac{1}{\rho}\exp(|\tau| x)\right), x < d, \\ (b_1 \cos\rho x + b_2 \cos\rho(2d - x)) + O\left(\frac{1}{\rho}\exp(|\tau| x)\right), x > d, \end{cases} \tag{2.4}$$

$$\varphi'(x, \lambda) = \begin{cases} -\rho \sin\rho x + O(\exp(|\tau| x)), x < d, \\ \rho(-b_1 \sin\rho x + b_2 \sin\rho(2d - x)) + O(\exp(|\tau| x)), x > d, \end{cases} \tag{2.5}$$

(2.6)

$$\psi(x, \lambda) = \begin{cases} b_1 \cos\rho(\pi - x) - b_2 \cos\rho(\pi + x - 2d) + O\left(\frac{1}{\rho}\exp(|\tau|(\pi - x))\right), x < d, \\ \cos\rho(\pi - x) + O\left(\frac{1}{\rho}\exp(|\tau|(\pi - x))\right), x > d, \end{cases}$$

(2.7)
$$\psi'(x,\lambda)=\begin{cases}\rho\left(b_1\sin\rho(\pi-x)+b_2\sin\rho(\pi+x-2d)\right)+O\left(\exp\left(|\tau|(\pi-x)\right)\right), x<d,\\ \rho\sin\rho(\pi-x)+O\left(\exp\left(|\tau|(\pi-x)\right)\right), x>d,\end{cases}$$

$$\Delta(\lambda)=\rho\left(b_1\sin\rho\pi-b_2\sin\rho(2d-\pi)\right)+O\left(\exp\left(|\tau|\pi\right)\right),$$

where $b_1=\dfrac{\beta+\beta^{-1}}{2}$, $b_2=\dfrac{\beta-\beta^{-1}}{2}$.

In particular, for $j=0,1$, we obtain

$$\varphi^{(j)}(x,\lambda)=O\left(|\rho|^{(j)}\exp\left(|\tau|x\right)\right), \tag{2.8}$$

$$\psi^{(j)}(x,\lambda)=O\left(|\rho|^{(j)}\exp\left(|\tau|(\pi-x)\right)\right). \tag{2.9}$$

Proof. The proof is similar to the selfadjoint case, see [25, p.145-146]. □

Lemma 3. *The roots* $\lambda_n^1=\left(\rho_n^1\right)^2$, $n\in\mathbb{N}$ *of*

$$\Delta^1(\lambda):=\rho\left(b_1\sin\rho\pi-b_2\sin\rho(2d-\pi)\right)$$

are separated. For fixed δ *and sufficiently large* $|\lambda|$,

$$\Delta(\lambda)\geqslant C_\delta|\rho|\exp\left(|\tau|\pi\right), \ \lambda\in G_\delta, \tag{2.10}$$

where $G_\delta=\left\{\lambda=\rho^2:\left|\rho-\rho_n^1\right|\geqslant\delta\right\}$. *By Rouché theorem, we have*

$$\rho_n=\sqrt{\lambda_n}=\rho_n^1+\frac{\theta_n}{\rho_n^1}+\frac{\kappa_n}{\rho_n^1},$$

so

$$\Delta(\lambda)=\varpi\left(\lambda-\lambda_0^1\right)\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{\lambda_n^1},$$

where $\varpi=\pi b_1-(2d-\pi)b_2$, $\kappa_n\in l_2$, *and* θ_n *is a bounded sequence*

$$\theta_n=\left(a_1\cos\rho_n^1\pi+a_2\cos\rho_n^1(2d-\pi)\right)\left(2\frac{d}{d\lambda}\Delta^1\left(\lambda_n^1\right)\right)^{-1},$$

$$a_1=b_1\left(h+H+\frac{1}{2}\int_0^\pi q(t)\,dt\right)+\frac{\gamma}{2},$$

$$a_2=b_2\left(H-h+\frac{1}{2}\int_0^\pi q(t)\,dt-\int_0^d q(t)\,dt\right)-\frac{\gamma}{2},$$

Proof. The proof can be refer to [25, p.146] and [2, Lemma 3]. □

3. The generalized spectral data

The algebraic multiplicity m_n of the eigenvalue $\lambda_n (n\in\mathbb{N})$ is the order of it as a root of $\Delta(\lambda)=0$, i.e. $\lambda_n=\lambda_{n+1}=\cdots=\lambda_{n+m_n-1}$. In throughout the paper, we use multiplicity instead of algebraic multiplicity for short. By the virtue of Lemma 3, for sufficient large n, $m_n=1$.

Let $S=\{n|n=1,2,\cdots,\ \lambda_{n-1}\neq\lambda_n\}\cup\{0\}$, $\varphi_\eta(x,\lambda)=\dfrac{1}{\eta!}\dfrac{d^\eta}{d\lambda^\eta}\varphi(x,\lambda)$, $\psi_\eta(x,\lambda)=\dfrac{1}{\eta!}\dfrac{d^\eta}{d\lambda^\eta}\psi(x,\lambda)$. For $\eta=1,2,\cdots,m_n-1$, $n\in S$, we have

$$\begin{cases}\ell\varphi_\eta(x,\lambda_n)=\lambda_n\varphi_\eta(x,\lambda_n)+\varphi_{\eta-1}(x,\lambda_n),\\ \varphi_\eta(d+0,\lambda_n)=\beta\varphi_\eta(d-0,\lambda_n),\\ \varphi'_\eta(d+0,\lambda_n)=\beta^{-1}\varphi'_\eta(d-0,\lambda_n)+\gamma\varphi_\eta(d-0,\lambda_n),\\ \varphi_\eta(0,\lambda_n)=\varphi'_\eta(0,\lambda_n)=0,\end{cases}\tag{3.1}$$

$$\begin{cases}\ell\psi_\eta(x,\lambda_n)=\lambda_n\psi_\eta(x,\lambda_n)+\psi_{\eta-1}(x,\lambda_n),\\ \psi_\eta(d+0,\lambda_n)=\beta\psi_\eta(d-0,\lambda_n),\\ \psi'_\eta(d+0,\lambda_n)=\beta^{-1}\psi'_\eta(d-0,\lambda_n)+\gamma\psi_\eta(d-0,\lambda_n),\\ \psi_\eta(\pi,\lambda_n)=\psi'_\eta(\pi,\lambda_n)=0.\end{cases}\tag{3.2}$$

From (2.3), we infer that

$$\frac{1}{\eta!}\Delta^{(\eta)}(\lambda_n)=-V(\varphi_\eta(x,\lambda_n))=U(\psi_\eta(x,\lambda_n))=0,\ n\in S,\ \eta=0,1,\cdots,m_n-1,$$

i.e., $\varphi_\eta(x,\lambda_n)$ and $\psi_\eta(x,\lambda_n)$, $n\in S$, $\eta=1,2,\cdots,m_n-1$, are generalized eigenfunctions of L. Let

$$\begin{array}{l}\varphi_{n+\eta}(x)=\varphi_\eta(x,\lambda_n),\ \psi_{n+\eta}(x)=\psi_\eta(x,\lambda_n),\\ \Delta_{\eta,n}:=\frac{1}{\eta!}\Delta^{(\eta)}(\lambda_n),\ n\in S,\eta=0,1,\cdots,m_n-1.\end{array}\tag{3.3}$$

It is easy to see that $\{\varphi_n(x)\}_{n\in\mathbb{N}}$, $\{\psi_n(x)\}_{n\in\mathbb{N}}$ are complete systems of eigenfunctions and generalized eigenfunctions of L (refer to [13, Theorem 1.3.2]). Naturally, we can define the generalized weight numbers α_n, $n\in\mathbb{N}$ for L by the following equations:

$$\alpha_{n+\eta}=\int_0^\pi\varphi_{n+\eta}(x)\varphi_{n+m_n-1}(x)\,dx,\ n\in S,\ \eta=0,1,\cdots,m_n-1.\tag{3.4}$$

When the multiplicity $m_n=1$, the generalized weight numbers α_n coincide with the weight numbers for the selfadjoint Sturm-Liouville operator with discontinuity conditions inside a finite interval (see [25, p.143 (10)]).

Definition 1. *The numbers $\{\lambda_n,\ \alpha_n\}_{n\in\mathbb{N}}$ are called the generalized spectral data of L.*

4. The Weyl function

Denote by $S(x,\lambda)$, $\Phi(x,\lambda)$ the solutions of equation (1.1) under the conditions

$$S'(0,\lambda)=U(\Phi)=1,\ S(0,\lambda)=V(\Phi)=0,$$

and the discontinuity conditions (1.3). The functions $\Phi(x,\lambda)$ and $M(\lambda):=\Phi(0,\lambda)$ are called the Weyl solution and the Weyl function for L, respectively. Evidently,

$$\Phi(x,\lambda)=\frac{\psi(x,\lambda)}{\Delta(\lambda)}=S(x,\lambda)+M(\lambda)\varphi(x,\lambda),\tag{4.1}$$

$$\langle\varphi(x,\lambda),\Phi(x,\lambda)\rangle\equiv1,\tag{4.2}$$

$$M(\lambda)=\frac{\Delta^0(\lambda)}{\Delta(\lambda)},\ \Delta^0(\lambda):=\psi(0,\lambda).\tag{4.3}$$

The symbol $\Delta^0(\lambda)$ denotes the characteristic function of the boundary value problem consisting of the equation (1.1), the discontinuity conditions (1.3) and the boundary conditions $y(0) = V(y) = 0$. The zeros of $\Delta^0(\lambda)$ are expressed in terms of $\left\{\lambda_n^0\right\}_{n\in\mathbb{N}}$, it is easy to show that $\left\{\lambda_n\right\}_{n\in\mathbb{N}} \cap \left\{\lambda_n^0\right\}_{n\in\mathbb{N}} = \emptyset$. Then $M(\lambda)$ is a meromorphic function with zeros in λ_n^0 and poles in λ_n.

Next, we prove that the generalized spectral data determine the Weyl function uniquely by the following theorem. This is a generalization of corresponding result of non-selfadjoint Sturm-Liouville operator without discontinuities (see [5, p.741 (9)]).

Theorem 1. *The Weyl function and the generalized spectral data of L satisfy the following equalities:*

$$M(\lambda) = \sum_{n\in S}\sum_{\eta=0}^{m_n-1} \frac{M_{n+\eta}}{(\lambda-\lambda_n)^{\eta+1}}, \tag{4.4}$$

$$\sum_{k=0}^{\eta} \alpha_{n+\eta-k} M_{n+m_n-k-1} = \delta_{\eta,0},\ n\in S,\ \eta = 0,1,\cdots,m_n-1, \tag{4.5}$$

where $\delta_{\eta,0}$ is Kronecker delta.

Proof. Firstly, considering the contour integral

$$I_N(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{M(\mu)}{\lambda-\mu}d\mu,\ \lambda\in \text{int}\ \Gamma_N,$$

where $\Gamma_N := \left\{\lambda : |\lambda| = R_N^2, R_N := \left|\rho_N^1\right| + \frac{1}{2}\inf_{\rho_M^1\neq\rho_N^1}\left|\rho_M^1-\rho_N^1\right|\right\}$, $M,\ N\in\mathbb{N}$, is assumed to be counterclockwise. By the virtue of Lemma 3, it yields $\Gamma_N\subset G_\delta$ for sufficiently small fixed $\delta>0$ and sufficiently large N. The formulae (2.9), (2.10), (4.3) yield

$$|M(\lambda)| \leqslant C|\rho|^{-1},\ \lambda\in G_\delta$$

for sufficiently large $|\lambda|$. Hence $\lim_{N\to\infty} I_N(\lambda) = 0$. By using the residue theorem (see [6, V. §2.]) we calculate

$$I_N(\lambda) = -M(\lambda) + \sum_{n\in S,\lambda_n\in\ \text{int}\ \Gamma_N} \operatorname*{Res}_{\mu=\lambda_n}\frac{M(\mu)}{\lambda-\mu}, \lambda\in int\Gamma_n\setminus\left\{\lambda_n\right\}_{n\in\mathbb{N}}.$$

Thus

$$M(\lambda) = \sum_{n\in S}\operatorname*{Res}_{\mu=\lambda_n}\frac{M(\mu)}{\lambda-\mu}. \tag{4.6}$$

Set $\operatorname*{Res}_{\mu=\lambda_n}\frac{M(\mu)}{\lambda-\mu} =: \sum_{\eta=0}^{m_n-1}\frac{M_{n+\eta}}{(\lambda-\lambda_n)^{\eta+1}}$, and in light of (4.6) we get (4.4).

Secondly, let us prove that coefficients M_n and the generalized weight numbers α_n determine each other uniquely by the formula (4.5). On account of (4.3) we have $M(\lambda)\Delta(\lambda) = \psi(0,\lambda)$, together with the identity (4.4) we find

$$\left(\sum_{n\in S}\sum_{\eta=0}^{m_n-1}\frac{M_{n+\eta}}{(\lambda-\lambda_n)^{\eta+1}}\right)\Delta(\lambda) = \psi(0,\lambda). \tag{4.7}$$

Since λ_n, $n\in S$, are the zeros of $\Delta(\lambda)$ with the multiplicity m_n, the Taylor series of $\Delta(\lambda)$ at λ_n, $n\in S$, is $\sum\limits_{p=m_n}^{\infty}\Delta_{p,n}(\lambda-\lambda_n)^p$. If we plug it back to (4.7) and let λ approaches λ_n, then $\psi_n(0)=M_{n+m_n-1}\Delta_{m_n,n}$. The proof of

$$\psi_{n+\eta}(0)=\sum_{k=0}^{\eta}M_{n+m_n-k-1}\Delta_{m_n+\eta-k,n}, n\in S,\ \eta=0,1,\cdots,m_n-1 \tag{4.8}$$

follows in a similar manner. From (4.1), we get $\psi_n(x)=\psi_n(0)\varphi_n(x)$, $n\in S$. Owing to (3.1)-(3.3), an easy induction gives

$$\psi_{n+\eta}(x)=\sum_{j=0}^{\eta}\psi_{n+j}(0)\varphi_{n+\eta-j}(x), n\in S,\ \eta=0,1,\cdots,m_n-1. \tag{4.9}$$

Moreover, since $\varphi(x,\lambda)$, $\psi(x,\mu)$ are solutions of equation (1.1) and satisfy the discontinuity conditions (1.3), from (2.1) we know the function $\langle y(x),z(x)\rangle$ is continuous on $x\in[0,\pi]$, hence

$$\frac{d}{dx}\langle\varphi(x,\lambda),\psi(x,\mu)\rangle=(\lambda-\mu)\varphi(x,\lambda)\psi(x,\mu).$$

By the initial conditions (2.2) and equality (2.3), we obtain

$$\frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu}=\int_0^\pi\varphi(x,\lambda)\psi(x,\mu)\,dx.$$

Hence $\frac{d}{d\lambda}\Delta(\lambda)=\int\limits_0^\pi\varphi(x,\lambda)\psi(x,\lambda)\,dx$. A simple manipulation leads to the solution that

$$\Delta_{m_n+\eta,n}=\frac{1}{m_n+\eta}\sum_{j=0}^{m_n+\eta-1}\int_0^\pi\varphi_{m_n+\eta-1-j}(x,\lambda_n)\psi_j(x,\lambda_n)\,dx,\ \eta\geqslant 0.$$

Using (3.1), (3.2) and integrating by parts we get

$$\Delta_{m_n+\eta,n}=\int_0^\pi\varphi_{n+m_n-1}(x)\psi_{n+\eta}(x)\,dx,\ n\in S,\ \eta=0,1,\cdots,m_n-1. \tag{4.10}$$

By substituting (4.9) in (4.10) and taking the definition of generalized weight numbers α_n (3.4) into account, we obtain

$$\Delta_{m_n+\eta,n}=\sum_{j=0}^{\eta}\alpha_{n+\eta-j}\psi_{n+j}(x). \tag{4.11}$$

Combining (4.11) and (4.8) we conclude that

$$\sum_{j=0}^{\eta}\psi_{n+\eta-j}(0)\sum_{k=0}^{j}\alpha_{n+j-k}M_{n+m_n-k-1}=\psi_{n+\eta}(0).$$

Since $\psi_n(0)\neq 0$, $n\in S$, continuing by induction we finally obtain the relation (4.5). $\square$

5. The inverse problem

Inverse Problem 1. *Recovering the operator L from one of the following conditions:* (i) *the generalized spectral data $\{\lambda_n, \alpha_n\}_{n\in\mathbb{N}}$;* (ii) *the two spectra $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{\lambda_n^0\}_{n\in\mathbb{N}}$;* (iii) *the Weyl function $M(\lambda)$.*

Remark 1. *According to Lemma 3, we know the spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$ uniquely determines the characteristic function $\Delta(\lambda)$. Similarly, the characteristic function $\Delta^0(\lambda)$ is uniquely determined by its zeros $\{\lambda_n^0\}_{n\in\mathbb{N}}$. Combining (4.3), (4.4) and (4.5), we see that the statements* (i)-(iii) *of Inverse Problems 1 are equivalent. The numbers $\{\lambda_n, M_n\}_{n\in\mathbb{N}}$ can also be used as spectral data.*

5.1. **The uniqueness theorem.** Before giving the uniqueness theorem, we introduce some symbols initially. We agree that L, $\tilde{L}$ denote the operators of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{d}$. That is to say if a certain symbol ξ represents an object related to L, then $\tilde{\xi}$ will denote the analogous object related to $\tilde{L}$, and $\hat{\xi} := \xi - \tilde{\xi}$.

Theorem 2. (*The uniqueness theorem*) *If $\lambda_n = \tilde{\lambda}_n$, $\alpha_n = \tilde{\alpha}_n$, $n \in \mathbb{N}$, then $L = \tilde{L}$, i.e. $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$, $h = \tilde{h}$, $H = \tilde{H}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $d = \tilde{d}$.*

Proof. Because of Theorem 1, we know the generalized spectral data $\{\lambda_n, \alpha_n\}_{n\in\mathbb{N}}$ uniquely determines the Weyl function $M(\lambda)$. It suffices to prove that if $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$. It follows from (2.9), (2.10) and (4.1) that

$$\left|\Phi^{(j)}(x, \lambda)\right| \leqslant C_\delta |\rho|^{j-1} \exp(-|\tau| x),\ j = 0, 1,\ \lambda \in G_\delta. \tag{5.1}$$

Define the matrix $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1.2}$ by the following formula

$$P(x, \lambda) \begin{bmatrix} \tilde{\varphi}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{\varphi}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x, \lambda) & \Phi(x, \lambda) \\ \varphi'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix},$$

i.e.,

$$\begin{cases} \varphi(x, \lambda) = P_{11}(x, \lambda) \tilde{\varphi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\varphi}'(x, \lambda), \\ \Phi(x, \lambda) = P_{11}(x, \lambda) \tilde{\Phi}(x, \lambda) + P_{12}(x, \lambda) \tilde{\Phi}'(x, \lambda). \end{cases} \tag{5.2}$$

Formula (4.2) yields

$$\begin{cases} P_{j,1}(x, \lambda) = \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}'(x, \lambda), \\ P_{j,2}(x, \lambda) = \Phi^{(j-1)}(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi^{(j-1)}(x, \lambda) \tilde{\Phi}(x, \lambda). \end{cases} \tag{5.3}$$

Combining (4.1) and (5.3) we see that

$$P_{11}(x, \lambda) = \varphi(x, \lambda) \tilde{S}'(x, \lambda) - S(x, \lambda) \tilde{\varphi}'(x, \lambda) + \left(\tilde{M}(\lambda) - M(\lambda)\right) \varphi(x, \lambda) \tilde{\varphi}'(x, \lambda),$$

$$P_{12}(x, \lambda) = S(x, \lambda) \tilde{\varphi}(x, \lambda) - \varphi(x, \lambda) \tilde{S}(x, \lambda) + \left(M(\lambda) - \tilde{M}(\lambda)\right) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda).$$

Owing to (4.1) and (5.3), for each fixed x, the functions $P_{jk}(x, \lambda)$ are meromorphic functions in λ. Put $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$. According to (2.8), (5.1) and (5.3), we obtain

$$|P_{12}(x, \lambda)| \leqslant C_\delta |\rho|^{-1},\ |P_{11}(x, \lambda)| \leqslant C_\delta,\ \lambda \in G_\delta^0. \tag{5.4}$$

By (4.1) and (5.3), we see that if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed x, the functions $P_{1k}(x, \lambda)$ are entire in λ. Combining with (5.4), we derive $P_{11}(x, \lambda) \equiv C(x)$, $P_{12}(x, \lambda) \equiv 0$. Taking (5.2) into consideration, we get

$$\varphi(x, \lambda) \equiv C(x) \tilde{\varphi}(x, \lambda) \tag{5.5}$$

for all x and λ. Together with (2.4), we see that for $|\rho| \to \infty$, $\arg\rho \in [\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$,

$$\varphi(x,\lambda) = \frac{b}{2}\exp(-i\rho x)\left(1 + O\left(\frac{1}{\rho}\right)\right),$$

where $b = 1$ for $x < d$, and $b = b_1$ for $x > d$. Combining (4.2) and (5.5) this yields $b_1 = \tilde{b}_1$, $C(x) \equiv 1$, i.e. $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ for all x and λ and consequently $L = \tilde{L}$. □

5.2. **Solution of the inverse problem.** Without loss of generality, we consider the inverse problem of recovering L from the generalized spectral data $\{\lambda_n,\ \alpha_n\}_{n\in\mathbb{N}}$. Like [25, p.153 (60)], choose an arbitrary model boundary value problem $\tilde{L} = \tilde{L}\left(\tilde{q}(x), \tilde{h}, \tilde{H}, \tilde{\beta}, \tilde{\gamma}, \tilde{d}\right)$ such that

$$d = \tilde{d},\ \left(\sum_{n=0}^{\infty}\left(\varsigma_n|\rho_n|\right)^2\right)^{1/2} < \infty, \sum_{n=0}^{\infty}\varsigma_n < \infty, \tag{5.6}$$

where $\varsigma_n := |\rho_n - \tilde{\rho}_n| + |\alpha_n - \tilde{\alpha}_n|$. Set $\lambda_{n,0} := \lambda_n$, $\lambda_{n,1} := \tilde{\lambda}_n$, $M_{n,0} := M_n$, $M_{n,1} := \tilde{M}_n$, $\varphi_{n,i}(x) := \varphi(x,\lambda_{n,i})$, $\tilde{\varphi}_{n,i}(x) := \tilde{\varphi}(x,\lambda_{n,i})$, $S_0 := S$, $S_1 := \tilde{S}$, $m_{n,0} := m_n$, $m_{n,1} := \tilde{m}_n$,

$$D(x,\lambda,\mu) := \frac{\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle}{\lambda-\mu}, D_{\eta,\nu}(x,\lambda,\mu) := \frac{1}{\eta!\nu!}\frac{\partial^{\eta+\nu}}{\partial\lambda^\eta\partial\mu^\nu}D(x,\lambda,\mu).$$

For $i,j = 0,1$, $n \in S_i$, denote

$$A_{n+\eta,i}(x,\lambda) := \sum_{p=\eta}^{m_{n,i}-1} M_{n+p,i}D_{0,p-\eta}(x,\lambda,\lambda_{n,i}),$$

$$Q_{n+\eta,i;k,j}(x) := \frac{1}{\eta!}\frac{\partial^\eta}{\partial\lambda^\eta}A_{k,j}(x,\lambda)|_{\lambda=\lambda_{n,i}},$$

where $k \in \mathbb{N}$, $\eta = 0,1,\cdots,m_{n,i}-1$. Similarly, by replacing φ with $\tilde{\varphi}$ in the above definitions we define $\tilde{D}(x,\lambda,\mu)$, $\tilde{D}_{\eta,\nu}(x,\lambda,\mu)$, $\tilde{A}_{n,i}(x,\lambda)$, $\tilde{Q}_{n,i;k,j}(x)$, $k \in \mathbb{N}$, $i,j = 0,1$. Using the fact that $\langle\varphi(x,\lambda),\varphi(x,\mu)\rangle$ is continuous on $x \in [0,\pi]$, $D(x,\lambda,\mu)$, $D_{\eta,\nu}(x,\lambda,\mu)$, $A_{n,i}(x,\lambda)$, $Q_{n,i;k,j}(x)$, $\tilde{D}(x,\lambda,\mu)$, $\tilde{D}_{\eta,\nu}(x,\lambda,\mu)$, $\tilde{A}_{n,i}(x,\lambda)$, $\tilde{Q}_{n,i;k,j}(x)$, $k \in \mathbb{N}$, $i,j = 0,1$ are continuous functions of $x \in [0,\pi]$.

By the same methods as in [25, p.153-156], using Lemma 2, Lemma 3, (2.1), (3.4), (4.5) and Schwarz's lemma [6, VI. §2.], we get the following estimates as $n,k \in \mathbb{N}$, $i,j = 0,1$:

$$\begin{cases} |\varphi_{n,i}(x)| \leqslant C, |\varphi_{n,0}(x) - \varphi_{n,1}(x)| \leqslant C\varsigma_n, |Q_{n,i;k,j}(x)| \leqslant \frac{C}{|\rho_n^1-\rho_k^1|+1}, \\ |Q_{n,i;k,0}(x) - Q_{n,i;k,1}(x)| \leqslant \frac{C\varsigma_k}{|\rho_n^1-\rho_k^1|+1}, \\ |Q_{n,0;k,j}(x) - Q_{n,1;k,j}(x)| \leqslant \frac{C\varsigma_n}{|\rho_n^1-\rho_k^1|+1}, \\ |Q_{n,0;k,0}(x) - Q_{n,1;k,0}(x) - Q_{n,0;k,1}(x) + Q_{n,1;k,1}(x)| \leqslant \frac{C\varsigma_n\varsigma_k}{|\rho_n^1-\rho_k^1|+1}, \end{cases} \tag{5.7}$$

The similar estimates are also valid for $\tilde{\varphi}_{n,i}(x)$, $\tilde{Q}_{n,i;k,j}(x)$.

Lemma 4. *The following representations hold:*

(5.8)

$$\tilde{\varphi}_{n,i}(x) = \varphi_{n,i}(x) + \sum_{k=0}^{\infty}\left(\tilde{Q}_{n,i;k,0}(x)\varphi_{k,0}(x) - \tilde{Q}_{n,i;k,1}(x)\varphi_{k,1}(x)\right),\ n \in \mathbb{N},\ i,j = 0,1,$$

(5.9)
$$\tilde{Q}_{n,i;k,j}(x) - Q_{n,i;k,j}(x) = \sum_{l=0}^{\infty}\left(\tilde{Q}_{n,i;l,0}(x)\,Q_{l,0;k,j}(x) - Q_{n,i;l,1}(x)\,\tilde{Q}_{l,1;k,j}(x)\right), \quad n,k\in\mathbb{N},\ i,j=0,1,$$

where the series converge absolutely and uniformly with respect to $x\in[0,\pi]$.

Proof. From (5.6), we obtain $d=\tilde{d}$ and $\beta=\tilde{\beta}$, then by virtue of (2.4), it yields

$$\left|\varphi^{(j)}(x,\lambda)-\tilde{\varphi}^{(j)}(x,\lambda)\right| \leqslant C\,|\rho|^{j-1}\exp\left(|\tau|\,x\right),\ j=0,1. \tag{5.10}$$

In the same way, we derive that

$$\left|\psi^{(j)}(x,\lambda)-\tilde{\psi}^{(j)}(x,\lambda)\right| \leqslant C\,|\rho|^{j-1}\exp\left(|\tau|\,(\pi-x)\right),\ j=0,1. \tag{5.11}$$

Let $G_\delta^0 = G_\delta\cap\tilde{G}_\delta$, using (2.6)-(2.7), (2.10), (4.1) and (5.11) we arrive at

$$\left|\Phi^{(j)}(x,\lambda)-\tilde{\Phi}^{(j)}(x,\lambda)\right| \leqslant C_\delta\,|\rho|^{j-2}\exp\left(-|\tau|\,x\right),\ j=0,1,\ \lambda\in G_\delta^0. \tag{5.12}$$

Further, combining (4.2) and (5.3), we see that

(5.13)
$$P_{11}(x,\lambda) = 1 + \left(\varphi(x,\lambda)-\tilde{\varphi}(x,\lambda)\right)\Phi'(x,\lambda) - \left(\Phi(x,\lambda)-\tilde{\Phi}(x,\lambda)\right)\tilde{\varphi}'(x,\lambda).$$

It follows from (2.8), (2.9), (5.1), (5.3), (5.10), (5.12) and (5.13) that

$$\left|P_{11}(x,\lambda)-1\right| \leqslant C_\delta\,|\rho|^{-1},\ \left|P_{12}(x,\lambda)\right| \leqslant C_\delta\,|\rho|^{-1},\ \lambda\in G_\delta^0. \tag{5.14}$$

Analogously, we have

$$\left|P_{22}(x,\lambda)-1\right| \leqslant C_\delta\,|\rho|^{-1},\ \left|P_{21}(x,\lambda)\right| \leqslant C_\delta,\ \lambda\in G_\delta^0. \tag{5.15}$$

Let real numbers a, b be $a<\min\mathrm{Re}\lambda_{n,i}$, $b>\max|\mathrm{Im}\lambda_{n,i}|$, $n\in\mathbb{N}$, $i=0,1$. Consider closed contour $\Upsilon_N := \partial\Omega_N$ (with counterclockwise circuit) in the λ-plane, where $\Omega_N := \left\{\lambda : a\leqslant\mathrm{Re}\lambda\leqslant R_N^2, |\mathrm{Im}\lambda|\leqslant b\right\}$. By the standard method (see [7, p.46-70]), using (4.1), (5.2)-(5.4), and Cauchy's integral formula (see [6, IV. §5.]), we obtain the identity

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \frac{1}{2\pi i}\int_{\Upsilon_N}\hat{M}(\mu)\,\tilde{D}(x,\lambda,\mu)\,\varphi(x,\mu)\,d\mu + \varepsilon_N(x,\lambda), \tag{5.16}$$

where

$$\varepsilon_N(x,\lambda) = \frac{1}{2\pi i}\int_{\Upsilon_N}\frac{\tilde{\varphi}(x,\lambda)\left(P_{11}(x,\mu)-1\right)+\tilde{\varphi}'(x,\lambda)\,P_{12}(x,\mu)}{\lambda-\mu}d\mu.$$

Using (5.14) we acquire

$$\lim_{N\to\infty}\frac{\partial^\eta}{\partial\lambda^\eta}\varepsilon_N(x,\lambda) = 0, \eta\geqslant 0$$

uniformly respect to $x\in[0,\pi]$ and λ on bounded sets. Similarly, we have the relation

(5.17)
$$\tilde{D}(x,\lambda,\mu) - D(x,\lambda,\mu) = \frac{1}{2\pi i}\int_{\Upsilon_N}\tilde{D}(x,\lambda,\xi)\,\hat{M}(\xi)\,D(x,\xi,\mu)\,d\xi + \varepsilon_N^1(x,\lambda,\mu),$$

where

$$\lim_{N\to\infty}\frac{\partial^{\eta+j}}{\partial\lambda^\eta\partial\mu^j}\varepsilon_N^1(x,\lambda,\mu) = 0, \eta, j\geqslant 0$$

uniformly with respect to $x \in [0,\pi]$ and λ, μ on bounded sets. Calculating the integral in (5.16) by the residue theorem (see [6, V. §2.]) we have, in light of (4.4),

$$\frac{1}{2\pi i}\int_{\Upsilon_N}\hat{M}(\mu)\,\tilde{D}(x,\lambda,\mu)\,\varphi(x,\mu)\,d\mu=\sum_{k=0}^{N}\left(\tilde{A}_{k,0}(x)\,\varphi_{k,0}(x)-\tilde{A}_{k,1}(x)\,\varphi_{k,1}(x)\right)$$

for sufficiently large N. Passing to the limit in (5.16) as $N\to\infty$ we obtain

$$\tilde{\varphi}(x,\lambda)=\varphi(x,\lambda)+\sum_{k=0}^{\infty}\left(\tilde{A}_{k,0}(x)\,\varphi_{k,0}(x)-\tilde{A}_{k,1}(x)\,\varphi_{k,1}(x)\right).$$

Taking derivative to the both sides of this equation with respect to λ the corresponding number of times and substituting into $\lambda=\lambda_{n,i}$, we arrive at (5.8). Analogously, using the same method on (5.17), it yields

$$\tilde{D}(x,\lambda,\mu)-D(x,\lambda,\mu)=\sum_{p=0}^{1}(-1)^p\sum_{l\in S_p}\sum_{\eta=0}^{m_{l,p}-1}D_{\eta,0}(x,\lambda_{l,p},\mu)\,\tilde{A}_{l+\eta,p}(x,\lambda),$$

and taking the definitions of $Q_{n,i;k,j}(x)$, $\tilde{Q}_{n,i;k,j}(x)$ into account we get (5.9). □

Note that there exists $N\in\mathbb{N}$, such that for $n>N$, $m_{n,0}=m_{n,1}=1$. Moreover, an argument similar to the one used in [27, Lemma 1.3.4] shows that the infinite series

$$\sum_{n=N+1}^{\infty}\left[M_{n,0}\tilde{\varphi}_{n,0}(x)\,\varphi_{n,0}(x)-M_{n,1}\tilde{\varphi}_{n,1}(x)\,\varphi_{n,1}(x)\right]$$

and

$$\sum_{n=N+1}^{\infty}\frac{d}{dx}\left[M_{n,0}\tilde{\varphi}_{n,0}(x)\,\varphi_{n,0}(x)-M_{n,1}\tilde{\varphi}_{n,1}(x)\,\varphi_{n,1}(x)\right]$$

converge absolutely and uniformly on $[0,d]$ and $[d,\pi]$, respectively. Therefore, $l(x):=-2l_0'(x)$ is square integrable on $[0,\pi]$, where

$$\begin{aligned}
l_0(x) \;&:=\; \sum_{n\in S_0}\sum_{\eta=0}^{m_{n,0}-1}\sum_{p=\eta}^{m_{n,0}-1}M_{n+p,0}\tilde{\varphi}_{n+p-\eta,0}(x)\,\varphi_{n+\eta,0}(x)\\
&\quad-\sum_{n\in S_1}\sum_{\eta=0}^{m_{n,1}-1}\sum_{p=\eta}^{m_{n,1}-1}M_{n+p,1}\tilde{\varphi}_{n+p-\eta,1}(x)\,\varphi_{n+\eta,1}(x)\\
&=\; \sum_{n\in S_0,n\leqslant N}\sum_{\eta=0}^{m_{n,0}-1}\sum_{p=\eta}^{m_{n,0}-1}M_{n+p,0}\tilde{\varphi}_{n+p-\eta,0}(x)\,\varphi_{n+\eta,0}(x)\\
&\quad-\sum_{n\in S_1,n\leqslant N}\sum_{\eta=0}^{m_{n,1}-1}\sum_{p=\eta}^{m_{n,1}-1}M_{n+p,1}\tilde{\varphi}_{n+p-\eta,1}(x)\,\varphi_{n+\eta,1}(x)\\
&\quad+\sum_{n=N+1}^{\infty}\left[M_{n,0}\tilde{\varphi}_{n,0}(x)\,\varphi_{n,0}(x)-M_{n,1}\tilde{\varphi}_{n,1}(x)\,\varphi_{n,1}(x)\right].
\end{aligned}$$

Lemma 5. *The following relations hold*

$$q(x)=\tilde{q}(x)+l(x),$$
$$\gamma=\left(\beta^{-1}-\beta^3\right)l_0(d-0)+\tilde{\gamma},$$
$$h=\tilde{h}-l_0(0),\,H=\tilde{H}+l_0(\pi).$$

Proof. The rigourous proof of this lemma is similar to [27, Lemma 1.3.5], [25, Lemma 5]. □

Remark 2. *For each fixed $x \in [0, \pi]$ the relation (5.8) can be considered as a system of linear equations with respect to $\varphi_{n,i}(x)$, $n \in \mathbb{N}$, $i = 0, 1$. But the series in (5.8) converges only "with brackets", i.e., the terms in them cannot be dissociated. Therefore, it is inconvenient to use (5.8) as a main equation of the inverse problem. Below we will transfer (5.8) to a linear equation in the Banach space of bounded sequences.*

Denote $\omega = \{u | u = (n, i), n \in \mathbb{N}, i = 0, 1\}$. For each fixed $x \in [0, \pi]$ we define the vector

$$\phi(x) = [\phi_u(x)]_{u \in \omega} = \begin{bmatrix} \phi_{n,0}(x) \\ \phi_{n,1}(x) \end{bmatrix}_{n \in \mathbb{N}}$$

by the formula

$$\begin{bmatrix} \phi_{n,0}(x) \\ \phi_{n,1}(x) \end{bmatrix} = \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{n,0}(x) \\ \varphi_{n,1}(x) \end{bmatrix}, \tag{5.18}$$

$$\chi_n = \begin{cases} \varsigma_n^{-1}, \varsigma_n \neq 0, \\ 0, \varsigma_n = 0. \end{cases}$$

We also define a block-matrix

$$H(x) = [H_{u;v}(x)]_{u,v \in \omega} = \begin{bmatrix} H_{n,0;k,0}(x) & H_{n,0;k,1}(x) \\ H_{n,1;k,0}(x) & H_{n,1;k,1}(x) \end{bmatrix}_{n,k \in \mathbb{N}}, \ u = (n, i), \ v = (k, j)$$

by the following formula

$$\begin{bmatrix} H_{n,0;k,0}(x) & H_{n,0;k,1}(x) \\ H_{n,1;k,0}(x) & H_{n,1;k,1}(x) \end{bmatrix} = \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{n,0;k,0}(x) & Q_{n,0;k,1}(x) \\ Q_{n,1;k,0}(x) & Q_{n,1;k,1}(x) \end{bmatrix} \begin{bmatrix} \varsigma_k & 1 \\ 0 & 1 \end{bmatrix}.$$

Similarly we introduce $\tilde{\phi}_{n,i}(x)$, $\tilde{\phi}(x)$ and $\tilde{H}_{n,i;k,j}(x)$, $\tilde{H}(x)$ by replacing $\varphi_{n,i}(x)$ by $\tilde{\varphi}_{n,i}(x)$, and $Q_{n,i;k,j}(x)$ by $\tilde{Q}_{n,i;k,j}(x)$. Using (5.7) we get the estimates

$$\begin{aligned} &|\phi_{n,i}(x)| \leqslant C, \ \left|\tilde{\phi}_{n,i}(x)\right| \leqslant C, \\ &|H_{n,i;k,j}(x)| \leqslant \tfrac{C\varsigma_k}{|\rho_n^1 - \rho_k^1| + 1}, \ \left|\tilde{H}_{n,i;k,j}(x)\right| \leqslant \tfrac{C\varsigma_k}{|\rho_n^1 - \rho_k^1| + 1}. \end{aligned} \tag{5.19}$$

Consider the Banach space B of bounded sequences $a = [a_u]_{u \in \omega}$ with the norm $\|a\|_B = \sup_{u \in \omega} |a_u|$. It follows from (5.19) that for each fixed $x \in [0, \pi]$ the operators $I + \tilde{H}(x)$ and $I - H(x)$ (here I is the identity operator), acting from B to B, are bounded, and

$$\begin{aligned} \|H(x)\|_{B \to B} &\leqslant C \sup \sum_{k=0}^{\infty} \frac{\varsigma_k}{|\rho_n^1 - \rho_k^1| + 1} < \infty, \\ \left\|\tilde{H}(x)\right\|_{B \to B} &\leqslant C \sup \sum_{k=0}^{\infty} \frac{\varsigma_k}{|\rho_n^1 - \rho_k^1| + 1} < \infty. \end{aligned}$$

Theorem 3. *For each fixed $x \in [0, \pi]$, the main equation*

$$\tilde{\phi}(x) = \left(I + \tilde{H}(x)\right) \phi(x) \tag{5.20}$$

for the vector $\phi(x) \in B$ is uniquely solvable in the Banach space B. Moreover, the operator $\left(I + \tilde{H}(x)\right)^{-1}$ is bounded in B.

Proof. Rewriting (5.8) in the form

$$\begin{bmatrix} \tilde{\varphi}_{n,0}(x) \\ \tilde{\varphi}_{n,1}(x) \end{bmatrix} = \begin{bmatrix} \varphi_{n,0}(x) \\ \varphi_{n,1}(x) \end{bmatrix} + \sum_{k=0}^{\infty} \begin{bmatrix} \tilde{Q}_{n,0;k,0}(x) & -\tilde{Q}_{n,0;k,1}(x) \\ \tilde{Q}_{n,1;k,0}(x) & -\tilde{Q}_{n,1;k,1}(x) \end{bmatrix} \begin{bmatrix} \varphi_{k,0}(x) \\ \varphi_{k,1}(x) \end{bmatrix}, \; n \in \mathbb{N},$$

substituting here (5.18) and taking into account our notations of $Q_{n,i;k,j}(x)$ and $\tilde{Q}_{n,i;k,j}(x)$ we arrive at

$$\tilde{\phi}_{n,i}(x) = \phi_{n,i}(x) + \sum_{k,j} \tilde{H}_{n,i;k,j}(x)\, \phi_{k,j}(x), \; (n,i), (k,j) \in \omega, \tag{5.21}$$

which is equivalent to (5.20) and the series in (5.21) converges absolutely and uniformly for $x \in [0,\pi]$. Similarly, by the definitions of $H_{n,i;k,j}(x)$, $\tilde{H}_{n,i;k,j}(x)$, (5.9) becomes

$$\tilde{H}_{n,i;k,j}(x) - H_{n,i;k,j}(x) = \sum_{l,p} \tilde{H}_{n,i;l,p}(x)\, H_{l,p;k,j}(x), \; (n,i), (k,j), (l,p) \in \omega,$$

which is equivalent to

$$\left(I + \tilde{H}(x)\right)(I - H(x)) = I.$$

Replacing L for $\tilde{L}$, one gets analogously

$$(I - H(x))\left(I + \tilde{H}(x)\right) = I.$$

Hence the operator $\left(I + \tilde{H}(x)\right)^{-1}$ exists, and it is bounded in B. □

Equation (5.20) is called the *main equation* of the inverse problem. Using the solution of the *main equation* one can construct the function q, the coefficients β, γ of the discontinuity conditions, and the coefficients h, H of the boundary conditions. Thus, we obtain the following algorithm for solving the inverse problem.

Algorithm 1. *Suppose the spectral data* $\{\lambda_n, \alpha_n\}_{n\in\mathbb{N}}$ *be given. Then*

(i) *calculate* M_n, $n \in \mathbb{N}$, *by solving the linear systems* (4.5);

(ii) *select* $\tilde{L} = \tilde{L}\left(\tilde{q}(x), \tilde{h}, \tilde{H}, \tilde{\beta}, \tilde{\gamma}, \tilde{d}\right)$ *satisfies* (5.6) *and calculate* $\tilde{\phi}(x)$ *and* $\tilde{H}(x)$;

(iii) *choose* $\phi(x)$ *by solving equation* (5.20) *and calculate* $\varphi_{n,0}(x)$ *via* (5.18);

(iv) *construct* q, γ, h, H, β *by Lemma* 5.

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