MATRIX REPRESENTATIONS FOR MULTIPLICATIVE NESTED SUMS

LIN JIU AND DIANE YAHUI SHI*

Abstract. We study the multiplicative nested sums, which are generalizations of the harmonic sums, and provide a calculation through multiplication of index matrices. Special cases interpret the index matrices as stochastic transition matrices of random walks on a finite number of sites. Relations among multiplicative nested sums, which are generalizations of relations between harmonic series and multiple zeta functions, can be easily derived from identities of the index matrices. Combinatorial identities and their generalizations can also be derived from this computation.

1. Introduction

The harmonic sums, defined by [BK99, eq. 4, p. 1]

$$S_{i_1,\ldots,i_k}(N) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} \frac{\operatorname{sign}(i_1)^{n_1}}{n_1^{|i_1|}} \times\cdots\times \frac{\operatorname{sign}(i_k)^{n_k}}{n_k^{|i_k|}}, \tag{1.1}$$

and [DMH17, p. 168]

$$H_{i_1,\ldots,i_k}(N) := \sum_{N> n_1>\cdots> n_k\geq 1} \frac{\operatorname{sign}(i_1)^{n_1}}{n_1^{|i_1|}} \times\cdots\times \frac{\operatorname{sign}(i_k)^{n_k}}{n_k^{|i_k|}}, \tag{1.2}$$

are naturally connected to zeta functions. For instance,
1. taking $k=1$, $i_1=x>0$ and $N\to\infty$, in either (1.1) or (1.2), gives the Riemann zeta-function $\zeta(x)$;
2. when $i_1,\ldots,i_k>0$ and $N\to\infty$, (1.2) becomes the multiple zeta value $\zeta(i_1,\ldots,i_k)$.

Applications of harmonic sums appear in various areas, such as [A12, p. 1] perturbative calculations of massless or massive single scale problems in quantum field theory. Ablinger [A12, Chpt. 6] implemented the `Mathematica` package `HarmonicSums.m`[1], based on the recurrence [B04, eq. 2.1, p. 21] that is inherited from the quasi-shuffle relations [H00, eq. 1, p. 51], for calculation of harmonic sums.

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*Corresponding Author.
[1]http://www.risc.jku.at/research/combinat/software/HarmonicSums/index.php

The current work here is to present an alternative calculation for, not only harmonic sums, but also for the general sums defined as follows.

Definition 1.1. We consider the *multiplicative nested sums* (MNS): for $m, N \in \mathbb{N}$,

$$S\left(f_1, \ldots, f_k; N, m\right) := \sum_{N \geq n_1 \geq \cdots \geq n_k \geq m} f_1\left(n_1\right) \cdots f_k\left(n_k\right), \tag{1.3}$$

and

$$A\left(f_1, \ldots, f_k; N, m\right) := \sum_{N > n_1 > \cdots > n_k \geq m} f_1\left(n_1\right) \cdots f_k\left(n_k\right). \tag{1.4}$$

That is, the summand is multiplicative $f\left(n_1, \ldots, n_k\right) = f_1\left(n_1\right) \cdots f_k\left(n_k\right)$, and the summation indices are nested. Here, for all $l = 1, \ldots, k$, f_l can be any function defined on $\{m, m+1, \ldots, N\}$, unless $N = \infty$ when convergence needs to be taken into consideration.

Remark. If $f_l(x) = \operatorname{sign}(i_l)^x / x^{|i_l|}$ for $l = 1, \ldots, k$, then (1.3) gives (1.1) and (1.4) gives (1.2).

In Section 2, we present the main theorem, i.e., the calculation for MNS, by associating to each function f_l an *index matrix* and then considering the multiplications. Since MNS are generalizations of harmonic sums, this method naturally works for harmonic sums. Rather than recursively applying the quasi-shuffle relations in [A12], this matrix calculation is more direct and also simultaneously calculate for multiple pairs of N and m. Properties of the index matrix, such as inverse, identities, and eigenvalues, eigenvectors, and diagonalization, follow after the main theorem.

Applications of this matrix calculation, presented in Section 3, connect different fields. Originally, this idea was inspired by constructing random walks for special harmonic sums. Different types of random walks appear in and connect to various fields. For example, the coefficients connecting Euler polynomials and generalized Euler polynomials [JMV14, eq. 3.8, p. 781] appear in a random walk over a finite number of sites [JMV14, Note 4.8, p. 787]. In Subsection 3.1, the special sum when $f_1 = \cdots = f_k = x^{-a}$ for $a \geq 1$ is interpreted as the probability of a certain random walk, while the index matrix is exactly the corresponding stochastic matrix.

Consider the limit case of harmonic sums (1.1) and (1.2), as $N \to \infty$ and further assuming $i_1, \ldots, i_k > 0$, i.e., $S\left(1/x^{i_1}, \ldots, 1/x^{i_k}; \infty, 1\right)$ and

$A\left(1/x^{i_1},\ldots,1/x^{i_k};\infty,1\right)$. The relation between them are of great importance and interest. For instance, the fact

$$S\left(\frac{1}{x^2},\frac{1}{x};\infty,1\right)=2A\left(\frac{1}{x^3};\infty,1\right)=2\zeta\left(3\right)$$

has been well studied and rediscovered many times. Hoffman [H92, Thm. 2.1, p. 277, Thm. 2.2, p. 278] obtained the symmetric summation expressions, in terms of the Riemann zeta-function ζ, for both $S(1/x^{i_1},\ldots,1/x^{i_k};\infty,1)$ and $A(1/x^{i_1},\ldots,1/x^{i_k};\infty,1)$. In particular, the direct relations, for $k=2$ and $k=3$, between $S\left(1/x^{i_1},\ldots,1/x^{i_k};\infty,1\right)$ and $A\left(1/x^{i_1},\ldots,1/x^{i_k};\infty,1\right)$ (see [H92, p. 276] or (3.5) and (3.6) below) are also easy to obtain. In Subsection 3.2, we provide the truncated and generalized version of these relations, easily derived from the identities of the index matrices.

Finally, we focus on combinatorial identities, where harmonic sums also appear. For instance, Dilcher [D95, Cor. 3, p. 93] established, for special harmonic sum, the relation

$$S_{\underbrace{1,\ldots,1}_{k}}\left(N\right)=\sum_{l=1}^{N}\binom{N}{l}\frac{\left(-1\right)^{l-1}}{l^k}, \tag{1.5}$$

from q-series of divisor functions. In Subsection 3.3, we present examples of combinatorial identities, including a generalization of (1.5). Here, those identities are obtained by applying calculations and properties of the index matrices. Therefore, all the examples can be viewed as alternative proofs.

2. Matrix representations and properties

In this section, we provide the matrix representations of MNS. The key is to associate to each function f_l an index matrix.

2.1. **Index matrices and calculation for MNS.**

Definition 2.1. Given a positive integer N and a function f on $\{1,\ldots,N\}$, we define the following $N\times N$ (lower triangular) *index matrices*:

$$\mathbf{S}_f:=\begin{pmatrix} f\left(1\right) & 0 & 0 & \cdots & 0\\ f\left(2\right) & f\left(2\right) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ f\left(N\right) & f\left(N\right) & f\left(N\right) & \cdots & f\left(N\right)\end{pmatrix} \tag{2.1}$$

and

$$\mathbf{A}_f := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ f(1) & 0 & 0 & \cdots & 0 & 0 \\ f(2) & f(2) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f(N-1) & f(N-1) & f(N-1) & \cdots & f(N-1) & 0 \end{pmatrix}. \tag{2.2}$$

Remarks. 1. Shifting $\mathbf{S}_f$ downward by one row gives $\mathbf{A}_f$, i.e.,

$$\mathbf{A}_f = (\delta_{i-1,j})_{N\times N}\, \mathbf{S}_f, \text{ where } \delta_{a,b} = \begin{cases} 1, & \text{if } a = b; \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

For simplicity, we further denote $\mathbf{\Delta} := (\delta_{i-1,j})_{N\times N}$ so that $\mathbf{A}_f = \mathbf{\Delta}\mathbf{S}_f$.
2. When the dimensions of index matrices need to be clarified, we use $\mathbf{S}_{N|f}$ and $\mathbf{A}_{N|f}$.

Through index matrices defined above, we could express MNS through matrix multiplications, as follows.

Theorem 2.2. *Let*

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{N\times N}.$$

Then, we have

$$S(f_1, \ldots, f_k; N, m) = \left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{S}_{f_l}\right)_{N,m}, \tag{2.4}$$

$$A(f_1, \ldots, f_k; N, m) = \left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{A}_{f_l}\right)_{N,m}, \tag{2.5}$$

where $\mathbf{M}_{i,j}$ denotes the entry located at the i^{th} row and j^{th} column of a matrix $\mathbf{M}$.

Proof. Since the proof for $A(f_1, \ldots, f_k; N, m)$ is similar, we shall only prove a stronger result for $S(f_1, \ldots, f_k; N, m)$, i.e., for integers $1 \le i, j \le N$

$$S(f_1, \ldots, f_k; i, j) = \left(\mathbf{P} \cdot \prod_{l=1}^{k} \mathbf{S}_{f_l}\right)_{i,j}.$$

1. When $k = 1$, it is easy to see that $(\mathbf{P} \cdot \mathbf{S}_{f_1})_{i,j} = \sum_{l=j}^{i} f_1(l) = S(f_1; i, j)$.

2. Suppose $S\left(f_1,\ldots,f_k;i,j\right)=\left(\mathbf{P}\cdot\prod_{l=1}^{k}\mathbf{S}_{f_l}\right)_{i,j}$. Then,

$$\begin{aligned}
S\left(f_1,\ldots,f_{k+1};i,j\right)&=\left(\mathbf{P}\cdot\prod_{l=1}^{k+1}\mathbf{S}_{f_l}\right)_{i,j}=\left(\left(\mathbf{P}\cdot\prod_{l=1}^{k}\mathbf{S}_{f_l}\right)\cdot\mathbf{S}_{f_{k+1}}\right)_{i,j}\\
&=\sum_{l=j}^{i}S\left(f_1,\ldots,f_k;i,l\right)f_{k+1}\left(l\right)\\
&=\sum_{l=j}^{i}f_{k+1}\left(l\right)\sum_{i\geq n_1\geq\cdots\geq n_k\geq l}f_1\left(n_1\right)\cdots f_k\left(n_k\right)\\
&=\sum_{i\geq n_1\geq\cdots\geq n_k\geq l\geq j}f_1\left(n_1\right)\cdots f_k\left(n_k\right)f_{k+1}\left(l\right)\\
&=S\left(f_1,\ldots,f_{k+1};i,j\right).
\end{aligned}$$

□

2.2. **Properties of the index matrix S.** Now, we study some properties of the index matrix. To simplify expressions in this section, we denote

$$\mathbf{S}_a:=\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0\\ a_2 & a_2 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_N & a_N & a_N & \cdots & a_N\end{pmatrix}$$

and $\mathbf{A}_a:=\mathbf{\Delta}\mathbf{S}_a$. In other words, we assume, in (2.1) and (2.2), $f(l)=a_l$ for all $l=1,\ldots,N$, so that we replace the lower index f by a. Next, we give some properties of $\mathbf{S}_a$.

Proposition 2.3. *Assume all $a_i\neq 0$.*
1. The inverse of $\mathbf{S}_a$ is given by

$$\mathbf{S}_a^{-1}=\begin{pmatrix} 1/a_1 & 0 & 0 & \cdots & 0 & 0\\ -1/a_1 & 1/a_2 & 0 & \cdots & 0 & 0\\ 0 & -1/a_2 & 1/a_3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1/a_{N-1} & 1/a_N\end{pmatrix}.$$

2. We have the matrix identities

$$\mathbf{S}_a^{-1}\mathbf{S}_{ab}\mathbf{S}_b^{-1}=\mathbf{I}-\mathbf{\Delta},\tag{2.6}$$

$$\mathbf{S}_a\mathbf{\Delta}\mathbf{S}_b\mathbf{\Delta}\mathbf{S}_c+\mathbf{S}_{ab}\mathbf{\Delta}\mathbf{S}_c+\mathbf{S}_a\mathbf{\Delta}\mathbf{S}_{bc}+\mathbf{S}_{abc}=\mathbf{S}_a\mathbf{S}_b\mathbf{S}_c.\tag{2.7}$$

3. $\mathbf{S}_a$ *has eigenvalues* $\{a_1, \ldots, a_N\}$*. If all the* a_j*'s are distinct, define* $\mathbf{D}_a = (d_{i,j})_{N\times N}$ *and* $\mathbf{E}_a := (e_{i,j})_{N\times N}$ *by*

$$d_{i,j} := \frac{a_i}{a_N}\prod_{k=i+1}^{N}\left(1-\frac{a_k}{a_j}\right) \text{ and } e_{i,j} := \frac{a_N}{a_i}\prod_{\substack{k=j\\k\neq i}}^{N}\frac{1}{1-\frac{a_k}{a_i}},$$

if $i \geq j$*, and* $d_{i,j} = 0 = e_{i,j}$ *otherwise. It follows that* $(d_{1,j}, \ldots, d_{N,j})^T$ *is an eigenvector of* $\mathbf{S}_a$*, with respect to* a_j*, and* $\mathbf{D}_a^{-1} = \mathbf{E}_a$*, implying*

$$\mathbf{S}_a = \mathbf{D}_a \operatorname{diag}(a_1, \ldots, a_N)\,\mathbf{E}_a, \tag{2.8}$$

where $\operatorname{diag}(a_1, \ldots, a_N)$ *means the diagonal matrix with entries* $\{a_1, \ldots, a_n\}$ *on the diagonal.*

Proof. We omit the straightforward computation and only sketch the idea here.
1. The inverse can be easily computed.
2. Denote $\mathbf{I}_a := \operatorname{diag}(1/a_1, \ldots, 1/a_N)$ and $\boldsymbol{\Delta}_a = \boldsymbol{\Delta}\mathbf{I}_a$ so that $\mathbf{S}_a^{-1} = \mathbf{I}_a - \boldsymbol{\Delta}_a$. Since $\mathbf{I}_a\mathbf{S}_{ab} = \mathbf{S}_b$ (but $\mathbf{S}_{ab}\mathbf{I}_b \neq \mathbf{S}_a$) and $\boldsymbol{\Delta}_a\mathbf{S}_{ab} = \mathbf{A}_b = \boldsymbol{\Delta}\mathbf{S}_b$, one easily obtains

$$\mathbf{S}_a^{-1}\mathbf{S}_{ab}\mathbf{S}_b^{-1} = (\mathbf{I}_a - \boldsymbol{\Delta}_a)\,\mathbf{S}_{ab}\mathbf{S}_b^{-1} = \mathbf{S}_b\mathbf{S}_b^{-1} - \boldsymbol{\Delta}\mathbf{S}_b\mathbf{S}_b^{-1} = \mathbf{I} - \boldsymbol{\Delta}.$$

Similarly, multiplying by $\mathbf{S}_a^{-1}$ from the left and by $\mathbf{S}_c^{-1}$ from the right on (2.7), we obtain

$$\boldsymbol{\Delta}\mathbf{S}_b\boldsymbol{\Delta} + \mathbf{S}_a^{-1}\mathbf{S}_{ab}\boldsymbol{\Delta} + \boldsymbol{\Delta}\mathbf{S}_{bc}\mathbf{S}_c^{-1} + \mathbf{S}_a^{-1}\mathbf{S}_{abc}\mathbf{S}_c^{-1} = \mathbf{S}_b,$$

which reduces to $\mathbf{I} - \boldsymbol{\Delta} = \mathbf{S}_b^{-1}\mathbf{S}_{bc}\mathbf{S}_c^{-1}$, i.e., (2.6).
3. The eigenvalues are easy to see. To verify the eigenvectors, it is equivalent to prove that for all $i = j, \ldots, N$, we have

$$\sum_{l=j}^{i} a_l \frac{a_i}{a_N}\prod_{k=l+1}^{N}\left(1-\frac{a_k}{a_j}\right) = a_j\frac{a_i}{a_N}\prod_{k=i+1}^{N}\left(1-\frac{a_k}{a_j}\right), \tag{2.9}$$

which can be directly computed by induction on i. The inverse $\mathbf{D}_a^{-1} = \mathbf{E}_a$ is equivalent to

$$\delta_{ij} = \sum_{t=j}^{i}\frac{a_i}{a_t}\left(\prod_{k=i+1}^{N}\left(1-\frac{a_k}{a_t}\right)\right)\cdot\left(\prod_{\substack{k=j\\k\neq t}}^{N}\frac{1}{1-\frac{a_k}{a_t}}\right), \tag{2.10}$$

which reduces to

$$\sum_{t=j}^{i}\left(\prod_{\substack{k=j\\k\neq t}}^{i}\frac{1}{a_t - a_k}\right) = \delta_{i,j}. \tag{2.11}$$

Now, consider the partial fraction decomposition [Z05, eq. 1, p. 313]

$$\frac{1}{(1-a_j z)\cdots(1-a_i z)}=\sum_{t=j}^{i}\frac{1}{1-a_t z}\left(\prod_{\substack{k=j\\k\neq t}}^{i}\frac{a_t}{a_t-a_l}\right).$$

By multiplying both sides by z and then letting $z\to\infty$, we obtain (2.11). Thus, the proof is complete. ☐

Remark. The diagonalization leads to an easy computation of powers of $\mathbf{S}_a$, namely,

$$(\mathbf{S}_a)^k=\mathbf{D}_a\,\mathrm{diag}\left(a_1^k,\ldots,a_N^k\right)\mathbf{E}_a. \tag{2.12}$$

3. Applications

Applications of the index matrix (2.1) and matrix expressions of MNS (2.4) appear in different areas. We shall study related topics in random walks, relations of harmonic series and multiple zeta values, and combinatorial identities, where one could see:
1. some index matrices are stochastic transition matrices of random walks;
2. identities of index matrices directly lead to relations among MNS, which are generalizations of that between harmonic series and multiple zeta values;
3. some combinatorial identities can be proven and generalized through matrix representations of MNS.

3.1. **Random walks.** In this subsection, we let $f_l(x)\equiv H_a(x):=1/x^a$ for $l=1,\ldots,k$, where $a\geq 1$. Assume $a=1$, then we have

$$\mathbf{S}_{H_1}=\begin{pmatrix}1&0&0&\cdots&0\\ \frac{1}{2}&\frac{1}{2}&0&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ \frac{1}{N}&\frac{1}{N}&\frac{1}{N}&\cdots&\frac{1}{N}\end{pmatrix}. \tag{3.1}$$

Now, label N sites as follows:

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \cdots\ \underset{N-1}{\bullet}\ \underset{N}{\bullet}$$

and consider a random walk starting from site "N", with the rules:

- one can only jump to sites that are NOT to the right of the current site, with equal probabilities;
- steps are independent.

Let $\mathbb{P}(i\to j)$ denote the probability from site "i" to site "j". For example, suppose we are at site "6":

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \underset{4}{\bullet}\ \underset{5}{\bullet}\ \overset{\text{here}}{\underset{6}{\bullet}}\ \underset{7}{\bullet}\ \underset{8}{\bullet}\ \cdots\ \underset{N}{\bullet}$$

then, the next step only allows to walk to sites $\{1,2,3,4,5,6\}$, with probabilities:

$$\mathbb{P}\left(6 \to 6\right) = \cdots = \mathbb{P}\left(6 \to 1\right) = \frac{1}{6}.$$

Therefore, a typical walk is as follows:

STEP 1: walk from "N" to some site "$n_1\left(\leq N\right)$", with $\mathbb{P}\left(N \to n_1\right) = \frac{1}{N}$;

STEP 2: walk from "n_1" to "$n_2\left(\leq n_1\right)$", with $\mathbb{P}\left(n_1 \to n_2\right) = \frac{1}{n_1}$;

$$\cdots \qquad \cdots$$

STEP $k+1$: walk $n_k \mapsto n_{k+1}\left(\leq n_k\right)$, with $\mathbb{P}\left(n_k \to n_{k+1}\right) = \frac{1}{n_k}$.

We consider the event that after $k+1$ steps, we arrive the site "m", i.e., $\mathbb{P}\left(n_{k+1} = m\right)$. Since the steps are independent,

(3.2)

$$\mathbb{P}\left(n_{k+1} = m\right) = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq m} \frac{1}{N n_1 \cdots n_k} = \frac{1}{N} S\left(\underbrace{H_1, \ldots, H_1}_{k}; N, m\right).$$

Meanwhile, the stochastic transition matrix is exactly given by $\mathbf{S}_{H_1}$, namely, $\mathbf{S}_{H_1} = \left(\mathbb{P}\left(i \to j\right)\right)_{N \times N}$. Thus,

$$\left(\left(\mathbf{S}_{H_1}\right)^{k+1}\right)_{N,m} = \mathbb{P}\left(n_{k+1} = m\right) = \frac{1}{N} S\left(\underbrace{H_1, \ldots, H_1}_{k}; N, m\right), \tag{3.3}$$

which is a probabilistic interpretation of (2.4) with the slight difference that $\mathbf{P}$ in (2.4) is replaced by $\mathbf{S}_{H_1}$.

As a result, this probabilistic interpretation of harmonic sums easily leads to the following limit.

Proposition 3.1.

$$\lim_{k \to \infty} S(\underbrace{H_1, H_1, \ldots, H_1}_{k}; N, 1) = \lim_{k \to \infty} \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1 \cdots n_k} = N.$$

Proof. From (3.2), we see that

$$S(\underbrace{H_1, H_1, \ldots, H_1}_{k}; N, 1) = N \cdot \mathbb{P}\left(n_{k+1} = 1\right).$$

Evidently, for a random walk among finite number of sites, the probability that it reaches the sink, i.e., eventually, this walk ends, is 1. More precisely, for each site $l = 2, \ldots, N$, the probability for the walk not moving to site 1 is

$$1 - \frac{1}{l} \leq 1 - \frac{1}{N}.$$

Thus,

$$\mathbb{P}\left(n_{k+1} \neq 1\right) \leq \left(1 - \frac{1}{N}\right)^{k+1} \to 0 \quad \text{as} \quad k \to \infty.$$

In other words,

$$\lim_{k\to\infty} \mathbb{P}\left(n_{k+1} = 1\right) = 1,$$

which completes the proof. $\square$

Remark. When $a > 1$, we could also form a similar random walk by

- adding another sink "R", to the right of "N", with $\mathbb{P}\left(R \to n\right) = \delta_{R,n}$ for $n \in \{1, \dots, N, R\}$;
- defining for $l = 1, 2, \dots, N$,

$$\mathbb{P}\left(l \to j\right) = \begin{cases} 0, & \text{if } l < j \le N; \\ \frac{1}{l^a}, & \text{if } 1 \le j \le l; \\ 1 - \frac{1}{l^{a-1}}, & \text{if } j = R. \end{cases}$$

Now for the stochastic transition matrix,

$$\begin{pmatrix} \mathbf{S}_a & * \\ * & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{S}_a & * \\ * & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} \mathbf{S}_a^{k+1} & * \\ * & 1 \end{pmatrix}$$

A similar calculation for $\mathbb{P}\left(n_{k+1} = 1\right)$ shows that

$$S\left(\underbrace{H_a, \dots, H_a}_{k}; N, m\right) = N^a \left(\left(\mathbf{S}_{H_a}\right)^{k+1}\right)_{N,m}. \tag{3.4}$$

The analogue of Prop. 3.1 fails, due to more than one sink, i.e.,

$$\lim_{k\to\infty} \mathbb{P}(n_{k+1} = 1) < 1.$$

3.2. **Relations between S and A.** Now, we consider the relations between the harmonic series $S\left(1/x^{i_1}, \dots, 1/x^{i_k}; \infty, 1\right)$ and the multiple zeta values $A\left(1/x^{i_1}, \dots, 1/x^{i_k}; \infty, 1\right)$. For example, when $k = 2$, we have

$$S\left(\frac{1}{x^{i_1}}, \frac{1}{x^{i_2}}; \infty, 1\right) = A\left(\frac{1}{x^{i_1}}, \frac{1}{x^{i_2}}; \infty, 1\right) + A\left(\frac{1}{x^{i_1+i_2}}; \infty, 1\right); \tag{3.5}$$

and when $k = 3$,

$$\begin{aligned} S\left(\frac{1}{x^{i_1}}, \frac{1}{x^{i_2}}, \frac{1}{x^{i_3}}; \infty, 1\right) =& A\left(\frac{1}{x^{i_1}}, \frac{1}{x^{i_2}}, \frac{1}{x^{i_3}}; \infty, 1\right) + A\left(\frac{1}{x^{i_1+i_2}}, \frac{1}{x^{i_3}}; \infty, 1\right) \\ &+ A\left(\frac{1}{x^{i_1}}, \frac{1}{x^{i_2+i_3}}; \infty, 1\right) + A\left(\frac{1}{x^{i_1+i_2+i_3}}; \infty, 1\right). \end{aligned} \tag{3.6}$$

Both the identities above can be found, e.g., in [H92, p. 276]. Next, we will establish the truncated and generalized versions of (3.5) and (3.6), in the sense that we truncate the series (from both above and below) into sums, which at the same time allows flexibility for general summands, not restricted to negative powers.

Theorem 3.2. *For positive integers N and m with $N > m$, we have*

$$S(f,g;N-1,m) = A(f,g;N,m) + A(fg;N,m) \tag{3.7}$$

and

$$\begin{aligned} S(f,g,h;N-1,m) =& A(f,g,h;N,m) + A(fg,h;N,m) \\ & + A(f,gh;N,m) + A(fgh;N,m)\,. \end{aligned} \tag{3.8}$$

Proof. By Theorem 2.2, the right-hand side of (3.7) is given by

$$(\mathbf{P}\mathbf{A}_f\mathbf{A}_g)_{N,m} + (\mathbf{P}\mathbf{A}_{fg})_{N,m} = (\mathbf{P}\boldsymbol{\Delta}(\mathbf{S}_f\boldsymbol{\Delta}\mathbf{S}_g + \mathbf{S}_{fg}))_{N,m}\,.$$

From (2.6), we see that

$$\mathbf{I} - \boldsymbol{\Delta} = (\mathbf{S}_f)^{-1}\mathbf{S}_{fg}(\mathbf{S}_g)^{-1} \Leftrightarrow \mathbf{S}_f\boldsymbol{\Delta}\mathbf{S}_g + \mathbf{S}_{fg} = \mathbf{S}_f\mathbf{S}_g.$$

Noticing the different dimensions, an easy observation shows that

$$\left(\mathbf{P}_N\boldsymbol{\Delta}_N\mathbf{S}_{N|f}\mathbf{S}_{N|g}\right)_{N,m} = \left(\mathbf{P}_{N-1}\mathbf{S}_{N-1|f}\mathbf{S}_{N-1|g}\right)_{N-1,m} = S(f,g;N-1,m)\,.$$

Similarly, (2.7) implies (3.8), by the replacement $(a,b,c) \mapsto (f,g,h)$. ☐

3.3. **Combinatorial identities.** The matrix computations in Section 2, especially the diagonalization for computing a matrix power, lead to alternative proofs for some combinatorial identities and their generalizations.

Example 3.3. Butler and Karasik [BK10, Thm. 4, p. 7] showed that if $G(n,k)$ satisfies $G(n,n) = 1$, $G(n,-k) = 0$ and for $k \geq 1$,

$$G(n,k) = G(n-1,k-1) + a_k G(n-1,k)\,,$$

then

$$S\left(\underbrace{a,\ldots,a}_{k},N,1\right) := \sum_{N\geq n_1\geq\cdots\geq n_k\geq 1} a_{n_1}\cdots a_{n_k} = G(N+k,N)\,,$$

based on a proof related to Stirling numbers of the second kind. Here, we provide an alternative proof without using Stirling numbers, as follows.

1. When $k = 1$, an induction on N shows directly that

$$\sum_{N\geq n_1\geq 1} a_{n_1} = a_N G(N,N) + G(N,N-1) = G(N+1,N)\,.$$

2. For the inductive step in k, similarly to (3.4), we see, by recurrence,

$$\begin{aligned} S\left(\underbrace{a,\ldots,a}_{k},N,1\right) &= a_N\left(\prod_{l=1}^{k}\mathbf{S}_a\right)_{N,1} = a_N\left(\mathbf{S}_a\left(\prod_{l=1}^{k-1}\mathbf{S}_a\right)\right)_{N,1} \\ &= \frac{1}{a_N}\sum_{m=1}^{N} a_N\cdot a_m G(m+k-1,m) \\ &= G(N+k,N)\,. \end{aligned}$$

Example 3.4. Suppose the $(a_m)_{m=1}^N$ are all distinct. An alternative expression of the previous example can be obtained by diagonalization.

$$\begin{aligned} S\left(\underbrace{a,\ldots,a}_{k},N,1\right) &= \frac{1}{a_N}\left(\mathbf{D}_{H_a}\operatorname{diag}\left\{a_1^{k+1},\ldots,a_N^{k+1}\right\}\mathbf{E}_{H_a}\right)_{N,1} \\ &= \frac{1}{a_N}\sum_{j=1}^{N} a_j^{k+1}\left(\frac{a_N}{a_j}\prod_{\substack{m=1\\ m\neq j}}^{N}\frac{1}{1-\frac{a_m}{a_j}}\right) \\ &= \sum_{j=1}^{N}\left(\prod_{\substack{m=1\\ m\neq j}}^{N}\frac{1}{1-\frac{a_m}{a_j}}\right)a_j^k. \end{aligned}$$

This recovers a general result [Z05, eq. 2, p. 313], which, when we take $a_j = (a-bq^{j+i-1})/(c-zq^{j+i-1})$ and $N = n-i+1$, "turns out to be a common source of several q-identities" [Z05, p. 314]. The special case $a_m = m^a$ yields

$$S_{\underbrace{a,\ldots,a}_{k}}(N) = \sum_{l=1}^{N}\left(\prod_{\substack{n=1\\ n\neq l}}^{N}\frac{n^a}{n^a-l^a}\right)\frac{1}{l^{ak}}, \tag{3.9}$$

which gives (1.5) when $a=1$.

Remark. When $a = m \in \mathbb{Z}$ and $m > 1$, consider the factorization

$$n^m - l^m = (n-l)\left(n-\xi_m l\right)\cdots\left(n-\xi_m^{m-1}l\right),$$

where $\xi_m := \exp(2\pi i/m)$, and $i^2 = -1$. We could obtain the following binomial-type expression

$$S_{\underbrace{a,\ldots,a}_{k}}(N) = \sum_{l=1}^{N}\left(\prod_{t=0}^{m-1}\binom{N}{\xi_m^t l}\frac{\pi\left(1-\xi_m^t\right)l}{\sin\left(\pi\xi_m^t l\right)}\right)\frac{1}{l^{mk}},$$

which is similar to (1.5) and the usual binomial coefficient is generalized as $\binom{x}{y} := \Gamma(x+1)/(\Gamma(x+1)\Gamma(x-y+1))$.

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Department of Mathematics and Statistics, Dalhousie University, 6316 Coburg Road, PO BOX 15000, Halifax, Nova Scotia, Canada B3H 4R2

E-mail address: `Lin.Jiu@dal.ca`

School of Mathematics, Tianjin University, No. 92 Weijin Road, Nankai District, Tianjin , P. R. China 300072

E-mail address: `shiyahui@tju.ed.cn`