THE THICKNESS OF THE COMPLETE MULTIPARTITE GRAPHS AND THE JOIN OF GRAPHS

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ABSTRACT. The thickness of a graph is the minimum number of planar spanning subgraphs into which the graph can be decomposed. It is known for relatively few classes of graphs, compared to other topological invariants, e.g., genus and crossing number. For the complete bipartite graphs, Beineke, Harary and Moon (On the thickness of the complete bipartite graph, *Proc. Cambridge Philos. Soc.*, 60 (1964), 1–5.) gave the answer for most graphs in this family in 1964. In this paper, we derive formulas and bounds for the thickness of some complete k-partite graphs. And some properties for the thickness for the join of two graphs are also obtained.

1. INTRODUCTION

A *graph* G is often denoted by $G = (V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set. The *complement* $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. A *complete graph* is a graph in which any two vertices are adjacent. A complete graph on n vertices is denoted by K_n. The *union* $G \cup H$ of two graph G and H is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. The *join* $G + H$ of two vertex disjoint graphs G and H is obtained from $G \cup H$ by joining every vertex of G to every vertex of H. A *complete k-partite graph* is a graph whose vertex set can be partitioned into k parts, such that every edge has its ends in different parts and every two vertices in different parts are adjacent. $K_{p_1,p_2,\ldots,p_k}$ denotes a complete k-partite graph in which the ith part contains p_i $(1 \le i \le k)$ vertices. And it is easy to see $K_{p_1,p_2,\ldots,p_k} = \overline{K}_{p_1} + \overline{K}_{p_2} + \ldots + \overline{K}_{p_k}$.

The *thickness* $t(G)$ of a graph G is the minimum number of planar spanning subgraphs into which G can be decomposed. It was firstly defined by Tutte [7] in 1963. Since determining the thickness of a graph is NP-hard [4], it is very difficult to get the exact thickness number for arbitrary graphs. Beineke and Harary [2] determined the thickness of the complete graph K_n for $n \equiv 4 (\mathrm{mod}\ 6)$ in 1965, while the remaining cases were solved in 1976, independently by Alekseev and Gončhakov [1] and by Vasak [8].

Theorem 1.1. [1, 2, 8] *The thickness of the complete graph K_n is $t(K_n) = \lfloor \frac{n+7}{6} \rfloor$, except that $t(K_9) = t(K_{10}) = 3$.*

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For the complete bipartite graphs, the problem has not been entirely solved yet. Beineke, Harary and Moon [3] gave the answer for most graphs in this family in 1964.

Theorem 1.2. [3] *For $m \leq n$, the thickness of the complete bipartite graph $K_{m,n}$ is $t(K_{m,n}) = \lceil \frac{mn}{2(m+n-2)} \rceil$, except possibly when m and n are both odd and there exists an integer k satisfying $m = \lfloor \frac{2k(m-2)}{(m-2k)} \rfloor$.*

For the complete tripartite graph, Poranen proved $t(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$ in [6] and Yang [9] gave the exact thickness number of $K_{l,m,n} (l \leq m \leq n)$ when $l + m \leq 5$ and showed that $t(K_{l,m,n}) = \lceil \frac{l+m}{2} \rceil$ when $l+m$ is even and $n > \frac{1}{2}(l+m-2)^2$; or $l+m$ is odd and $n > (l + m - 2)(l + m - 1)$. The reader is referred to [5] for more background and results about the thickness problems.

In this paper, our concerning is the thickness of the complete multipartite graphs. Some results on the thickness for the join of two graphs are also given.

2. Thickness of some complete k-partite graphs

The *arboricity* $a(G)$ of a graph G is the minimum number of spanning forests into which the graph can be decomposed. It is known that the thickness of any graph is not less than a third of its arboricity and not more than its arboricity, and the arboricity of the complete graph K_n is $\lceil \frac{n}{2} \rceil$.

Lemma 2.1. *Let $K_s + \overline{K}_n$ be the join of K_s and $\overline{K}_n$ $(s, n \geq 1)$, then*

$$t(K_s + \overline{K}_n) \ \leq \ \lceil \frac{s}{2} \rceil.$$

Proof. Denote the s vertices in K_s by $v_1, \ldots, v_s$, and the n vertices in $\overline{K}_n$ by $u_1, \ldots, u_n$. In the following, we will construct a planar subgraphs decomposition of $K_s + \overline{K}_n$ with $\lceil \frac{s}{2} \rceil$ planar subgraphs, which shows $t(K_s + \overline{K}_n) \leq \lceil \frac{s}{2} \rceil$.

(1) Since $a(K_s) = \lceil \frac{s}{2} \rceil$, we have a spanning forests decomposition of K_s with $\lceil \frac{s}{2} \rceil$ forests, and we denote these forests by $F_1, \ldots, F_{\lceil \frac{s}{2} \rceil}$.
(2) Add n parallel edges between v_1 and v_2 in F_1 and insert vertices $u_1, \ldots, u_n$ on these n parallel edges respectively. We will get a planar subgraph G_1 of $K_s + \overline{K}_n$.
(3) Add n parallel edges between v_3 and v_4 in F_2 and insert vertices $u_1, \ldots, u_n$ on these n parallel edges respectively. We will get a planar subgraph G_2 of $K_s + \overline{K}_n$.
(4) Repeat this procedure with F_i, for $3 \leq i \leq \lceil \frac{s}{2} \rceil$. If s is odd, place the vertices $u_1, \ldots, u_n$ in $F_{\lceil \frac{s}{2} \rceil}$ and join them to v_s. We will get planar subgraphs $G_3, \ldots, G_{\lceil \frac{s}{2} \rceil}$ of $K_s + \overline{K}_n$.

For $G_1 \cup G_2 \cup \cdots \cup G_{\lceil \frac{s}{2} \rceil} = K_s + \overline{K}_n$, a planar subgraphs decomposition of $K_s + \overline{K}_n$ with $\lceil \frac{s}{2} \rceil$ planar subgraphs is obtained, and the lemma follows. $\square$

Lemma 2.2. *For the complete k-partite graph $K_{p_1,p_2,\ldots,p_k}$ $(k \geq 3)$, let s be the number of odd numbers in the set $\{p_1, p_2, \ldots, p_{k-1}\}$, then*

$$t(K_{p_1,p_2,\ldots,p_k}) \leq \sum_{i=1}^{k-1} \left\lfloor \frac{p_i}{2} \right\rfloor + \left\lceil \frac{s}{2} \right\rceil .$$

Proof. Suppose the k partite sets of $K_{p_1,p_2,\ldots,p_k}$ are $V_1 = \{v_1^1, v_2^1, \ldots, v_{p_1}^1\}$, $V_2 = \{v_1^2, v_2^2, \ldots, v_{p_2}^2\}$, ..., $V_k = \{v_1^k, v_2^k, \ldots, v_{p_k}^k\}$ respectively. We will construct one of its planar subgraphs decomposition as follows.

(1) For vertices in V_1, join both v_1^1 and v_2^1 to all the vertices in V_i, $1 \leq i \leq k$ and $i \neq 1$, we can get a planar graph G_1^1. Join both v_3^1 and v_4^1 to all the vertices in V_i, $1 \leq i \leq k$ and $i \neq 1$, we can get a planar graph G_2^1. Repeat this procedure with different vertices from V_1, until the $\lfloor \frac{p_1}{2} \rfloor$th planar graph $G^1_{\lfloor \frac{p_1}{2} \rfloor}$ has been obtained. If p_1 is even, then all the vertices from V_1 will be used. If p_1 is odd, then the vertex $v_{p_1}^1$ will not be used.

(2) For vertices in V_2, join both v_1^2 and v_2^2 to all the vertices in V_i, $1 \leq i \leq k$ and $i \neq 2$, we can get a planar graph G_1^2. Join both v_3^2 and v_4^2 to all the vertices in V_i, $1 \leq i \leq k$ and $i \neq 2$, we can get a planar graph G_2^2. Repeat this procedure with different vertices from V_2, until the $\lfloor \frac{p_2}{2} \rfloor$th planar graph $G^2_{\lfloor \frac{p_2}{2} \rfloor}$ has been obtained.

(3) Repeat this procedure with $V_3, \ldots, V_{k-1}$ respectively, we will get $\lfloor \frac{p_3}{2} \rfloor + \cdots + \lfloor \frac{p_{k-1}}{2} \rfloor$ planar subgraphs of $K_{p_1,p_2,\ldots,p_k}$, denote them by $G_1^3, \ldots, G^3_{\lfloor \frac{p_3}{2} \rfloor}$, ..., $G_1^{p_{k-1}}, \ldots, G^{p_{k-1}}_{\lfloor \frac{p_{k-1}}{2} \rfloor}$ respectively.

(4) Let

$$G = G_1^1 \cup \cdots \cup G^1_{\lfloor \frac{p_1}{2} \rfloor} \cup \cdots \cup G_1^{p_{k-1}} \cup \cdots \cup G^{p_{k-1}}_{\lfloor \frac{p_{k-1}}{2} \rfloor},$$

then $K_{p_1,p_2,\ldots,p_k} - G = K_s + \overline{K}_{p_k}$, in which s is the number of odd numbers in the set $\{p_1, p_2, \ldots, p_{k-1}\}$. By Lemma 2.1, there exists a planar subgraphs decomposition of $K_s + \overline{K}_{p_k}$ with $\lceil \frac{s}{2} \rceil$ subgraphs.

Above all, a planar decomposition of $K_{p_1,p_2,\ldots,p_k}$ with $\displaystyle\sum_{i=1}^{k-1} \left\lfloor \frac{p_i}{2} \right\rfloor + \left\lceil \frac{s}{2} \right\rceil$ planar subgraphs is obtained, and the lemma follows. □

Theorem 2.3. *The thickness of the complete k-partite graph $K_{p_1,p_2,\ldots,p_k}$ $(k \geq 3)$ equals $\left\lceil \displaystyle\sum_{i=1}^{k-1} \frac{p_i}{2} \right\rceil$ when $\displaystyle\sum_{i=1}^{k-1} p_i$ is even and $p_k > \dfrac{1}{2} \left(\displaystyle\sum_{i=1}^{k-1} p_i - 2 \right)^2$, or $\displaystyle\sum_{i=1}^{k-1} p_i$ is odd and $p_k > \left(\displaystyle\sum_{i=1}^{k-1} p_i - 1 \right) \left(\displaystyle\sum_{i=1}^{k-1} p_i - 2 \right)$.*

Proof. When $\sum_{i=1}^{k-1} p_i$ is even and $p_k > \frac{1}{2}\left(\sum_{i=1}^{k-1} p_i - 2\right)^2$, or $\sum_{i=1}^{k-1} p_i$ is odd and $p_k > \left(\sum_{i=1}^{k-1} p_i - 1\right)\left(\sum_{i=1}^{k-1} p_i - 2\right)$, from [3], the thickness of the complete bipartite graph $K_{p_1+p_2+\ldots+p_{k-1},p_k}$ is $\left\lceil \sum_{i=1}^{k-1} \frac{p_i}{2} \right\rceil$. Since $K_{p_1+p_2+\ldots+p_{k-1},p_k}$ is a subgraph of $K_{p_1,p_2,\ldots,p_k}$, by Lemma 2.2, we have

$$\sum_{i=1}^{k-1} \left\lfloor \frac{p_i}{2} \right\rfloor + \left\lceil \frac{s}{2} \right\rceil \geq t(K_{p_1,p_2,\ldots,p_k}) \geq t(K_{p_1+p_2+\cdots+p_{k-1},p_k}) = \left\lceil \sum_{i=1}^{k-1} \frac{p_i}{2} \right\rceil,$$

in which s is the number of odd numbers in the set $\{p_1, p_2, \ldots, p_{k-1}\}$. Since

$$\sum_{i=1}^{k-1} \left\lfloor \frac{p_i}{2} \right\rfloor + \left\lceil \frac{s}{2} \right\rceil = \left\lceil \sum_{i=1}^{k-1} \frac{p_i}{2} \right\rceil$$

where $0 \leq s \leq k-1$, the theorem is obtained. □

3. The thickness of $G + \overline{K}_n$

Let $p_1 = p_2 = \cdots = p_s = 1$ and $p_{s+1} = n$, from Theorem 2.3, we have the following theorem.

Theorem 3.1. *The thickness of the join of K_s and $\overline{K}_n$ $(s, n \geq 1)$ equals $\lceil \frac{s}{2} \rceil$, if s is even and $n > \frac{1}{2}(s-2)^2$, or s is odd and $n > (s-1)(s-2)$.*

Theorem 3.2. *If G is a simple graph on p vertices, then the thickness of $G+\overline{K}_n$ is $\lceil \frac{p}{2} \rceil$ when p is even and $n > \frac{1}{2}(p-2)^2$, or p is odd and $n > (p-1)(p-2)$.*

Proof. Since $K_{p,n} \subseteq G + \overline{K}_n \subseteq K_p + \overline{K}_n$, we have $t(K_{p,n}) \leq t(G + \overline{K}_n) \leq t(K_p + \overline{K}_n)$. From Theorem 3.1 and the Theorem 1 in [3], the theorem can be obtained. □

Theorem 3.3. *The thickness of the join of K_s and $\overline{K}_n$ $(s \leq 4, n \geq 1)$ equals*

$$t(K_s + \overline{K}_n) = \begin{cases} 1, & \text{if } s = 1, 2 \text{ or } s = 3 \text{ and } n \leq 2; \\ 2, & \text{if } s = 3 \text{ and } n \geq 3, \text{ or } s = 4 \text{ and } n \geq 1. \end{cases}$$

Proof. It is easy to see that the graphs $K_1 + \overline{K}_n$, $K_2 + \overline{K}_n$, $K_3 + \overline{K}_1$ and $K_3 + \overline{K}_2$ are all planar graphs, so their thicknesses are all one.

When $n \geq 3$, the graph $K_3 + \overline{K}_n$ contains a $K_{3,3}$ as its subgraph, so we have $t(K_3 + \overline{K}_n) \geq 2$. On the other hand, it is trivial to construct a planar subgraphs decomposition of $K_3 + \overline{K}_n$ with two planar subgraphs, using the construction of Lemma 2.1, which shows $t(K_3 + \overline{K}_n) \leq 2$. Hence, $t(K_3 + \overline{K}_n) = 2$ when $n \geq 3$.

When $n \geq 1$, the graph $K_4 + \overline{K}_n$ contains a K_5 as its subgraph, so we have $t(K_4+\overline{K}_n) \geq 2$. And it is not hard to construct a planar subgraphs decomposition of $K_4 + \overline{K}_n$ with two planar subgraphs, which shows $t(K_4 + \overline{K}_n) \leq 2$. Hence, $t(K_4 + \overline{K}_n) = 2$ when $n \geq 1$.

Summarizing the above, the theorem follows. $\square$

The following result shows that the upper bound in Theorem 3.1 is best possible for $s = 5$.

Corollary 3.4. *When $1 \leq n \leq 12$, the thickness of $K_5 + \overline{K}_n$ is 2; when $n \geq 13$, the thickness of $K_5 + \overline{K}_n$ is 3.*

Proof. From Theorem 3.1, the thickness of $K_5 + \overline{K}_n$ is 3, when $n \geq 13$. Because the graph $K_5 + \overline{K}_n$ contains a K_5 as its subgraph, we have $t(K_5 + \overline{K}_n) \geq 2$. Figure 1 constructs a planar subgraphs decomposition of $K_5 + \overline{K}_{12}$ with two planar subgraphs, which shows $t(K_5 + \overline{K}_n) \leq 2$, when $1 \leq n \leq 12$. Summarizing the above, the corollary follows. $\square$

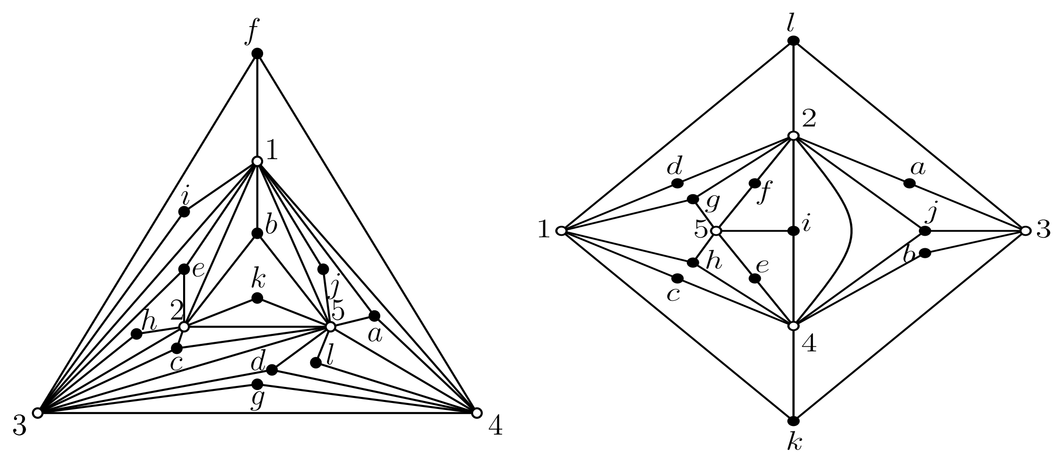


FIGURE 1. A planar decomposition of $K_5 + \overline{K}_{12}$

Though we do not find a formula of $t(K_s + \overline{K}_n)$, for all s and n, we have the following upper and lower bounds.

Corollary 3.5. *For $s, n \geq 1$ and $s \notin \{8, 9\}$, the possible thicknesses of $K_s + \overline{K}_n$ are consecutive integers in $[\lfloor \frac{s+8}{6} \rfloor, \lceil \frac{s}{2} \rceil]$.*

Proof. Because $t(K_s + \overline{K}_n) \geq t(K_s + \overline{K}_1)$ and $K_s + \overline{K}_1 = K_{s+1}$, combining it with Theorem 1.1, we have $t(K_s + \overline{K}_n) \geq \lfloor \frac{s+8}{6} \rfloor$, except for $s = 8$ and 9. From Theorem 3.1, the maximum value of $t(K_s + \overline{K}_n)$ equals $\lceil \frac{s}{2} \rceil$. Since the graph $K_s + \overline{K}_{n+1}$ is obtained from $K_s + \overline{K}_n$ by adding a new vertex and joining the new vertex to all vertices in K_s, we have $t(K_s + \overline{K}_{n+1}) = t(K_s + \overline{K}_n)$ or $t(K_s + \overline{K}_{n+1}) = t(K_s + \overline{K}_n) + 1$. Summarizing the above, the corollary is obtained. $\square$

In a similar way, the following two corollaries can be obtained.

Corollary 3.6. *For $n \geq 1$, the thickness of $K_8 + \overline{K}_n$ is 3 or 4.*

Corollary 3.7. *For $n \geq 1$, the thickness of $K_9 + \overline{K}_n$ is 3, 4 or 5.*

4. The thickness of $G + P_n$

The graph P_n is the path with n vertices. In this section we study the thickness of the join of a graph G and a path P_n.

Theorem 4.1. *The thickness of the join of K_s and P_n $(s, n \geq 1)$ equals $\lceil \frac{s}{2} \rceil$, if s is even and $n > \frac{1}{2}(s-2)^2$, or s is odd and $n > (s-1)(s-2)$.*

Proof. In the proof of Lemma 2.1, we have constructed a planar subgraphs decomposition of $K_s + \overline{K}_n$ with $\lceil \frac{s}{2} \rceil$ planar subgraphs by the procedure (1)-(4). By joining the vertices $u_1, u_2, \ldots, u_n$ with a path in the process (2), we can get a planar subgraphs decomposition of $K_s + P_n$ with $\lceil \frac{s}{2} \rceil$ planar subgraphs, which shows $t(K_s + P_n) \leq \lceil \frac{s}{2} \rceil$. On the other hand, $K_s + \overline{K}_n$ is a subgraph of $K_s + P_n$, so we have $t(K_s + \overline{K}_n) \leq t(K_s + P_n)$. Combining them with Theorem 3.1, the theorem follows. □

Theorem 4.2. *If G is a simple graph on s vertices, then the thickness of $G + P_n$ is $\lceil \frac{s}{2} \rceil$ when s is even and $n > \frac{1}{2}(s-2)^2$, or s is odd and $n > (s-1)(s-2)$.*

Proof. With a similar proof to that of Theorem 3.2, the theorem can be obtained. □

Theorem 4.3. *The thickness of the join of K_s and P_n $(s \leq 5, n \geq 1)$ equals*

$$t(K_s + P_n) = \begin{cases} 1, & \text{if } s = 1, 2 \text{ or } s = 3 \text{ and } n = 1; \\ 2, & \text{if } s = 3 \text{ and } n \geq 2, \text{ or } s = 4, \text{ or } s = 5 \text{ and } 1 \leq n \leq 12; \\ 3, & \text{if } s = 5 \text{ and } n \geq 13. \end{cases}$$

Proof. It is easy to see that the graphs $K_1 + P_n$, $K_2 + P_n$ and $K_3 + P_1$ are all planar graphs, so their thicknesses are all one.

When $n \geq 2$, the graph $K_3 + P_n$ contains a K_5 as its subgraph, so we have $t(K_3 + P_n) \geq 2$. However, Figure 2 illustrates a planar subgraphs decomposition of $K_3 + P_n$ with two planar subgraphs, which shows $t(K_3 + P_n) \leq 2$. So we have $t(K_3 + P_n) = 2$ when $n \geq 2$.

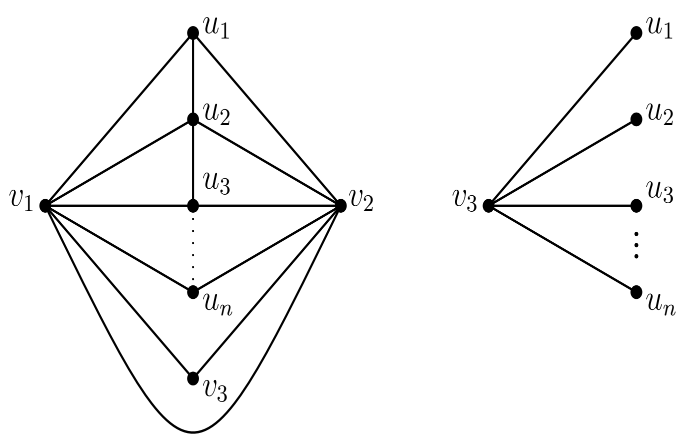


Figure 2. A planar decomposition of $K_3 + P_n$

When $n \geq 1$, the graph $K_4 + P_n$ contains a K_5 as its subgraph, so we have $t(K_4 + P_n) \geq 2$. And Figure 3 presents a planar subgraphs decomposition of $K_4 + P_n$ with two planar subgraphs, which shows $t(K_4 + P_n) \leq 2$. So we have $t(K_4 + P_n) = 2$ when $n \geq 1$.

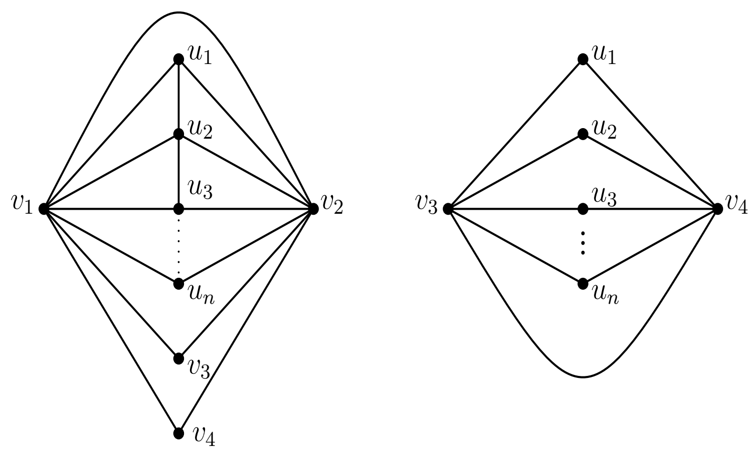


FIGURE 3. A planar decomposition of $K_4 + P_n$

From Theorem 4.1, the thickness of $K_5 + P_n$ is three, when $n \geq 13$. Because the graph $K_5 + P_n$ contains a K_5 as its subgraph, we have $t(K_5 + P_n) \geq 2$. Figure 4 constructs a planar subgraphs decomposition of $K_5 + P_{12}$ with two planar subgraphs, which shows $t(K_5 + P_n) \leq 2$, when $1 \leq n \leq 12$. Hence the thickness of $K_5 + P_n$ is two, when $1 \leq n \leq 12$.

Summarizing the above, the theorem follows. □

Proceeding as the proof of Corollary 3.5, we have the following corollaries.

Corollary 4.4. *For $s, n \geq 1$ and $s \notin \{8, 9\}$, the possible thicknesses of $K_s + P_n$ are consecutive integers in $[\lfloor \frac{s+8}{6} \rfloor, \lceil \frac{s}{2} \rceil]$.*

Corollary 4.5. *For $n \geq 1$, the thickness of $K_8 + P_n$ is 3 or 4.*

Corollary 4.6. *For $n \geq 1$, the thickness of $K_9 + P_n$ is 3, 4 or 5.*

5. The thickness for the join of two graphs G and H

For arbitrary graphs G and H, we prove the following two properties for the thickness of the join of G and H.

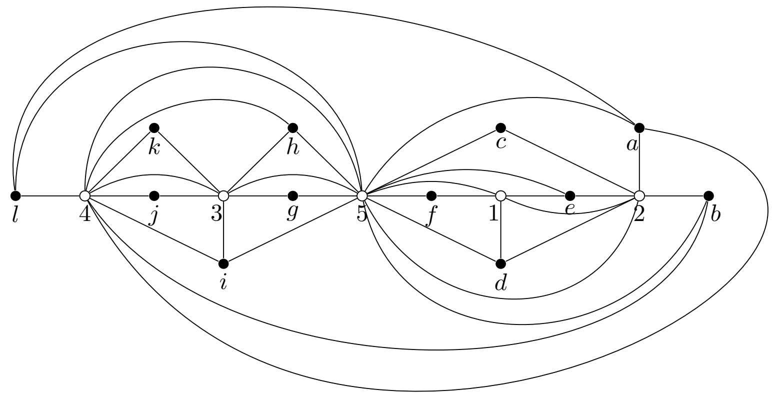
Property 5.1. *Let G and H be two simple graphs with order p and q respectively, then*

$$t(G + H) \leq Max\{t(G), t(H)\} + t(K_{p,q}).$$

Proof. Because $G + H$ is a subgraph of $G \cup H \cup K_{p,q}$, in which the two partite sets of $K_{p,q}$ are $V(G)$ and $V(H)$ respectively,

$$t(G + H) \leq t(G \cup H \cup K_{p,q}) \leq t(G \cup H) + t(K_{p,q}).$$

Since G and H are disjoint, $t(G \cup H) = Max\{t(G), t(H)\}$, and the property follows. □



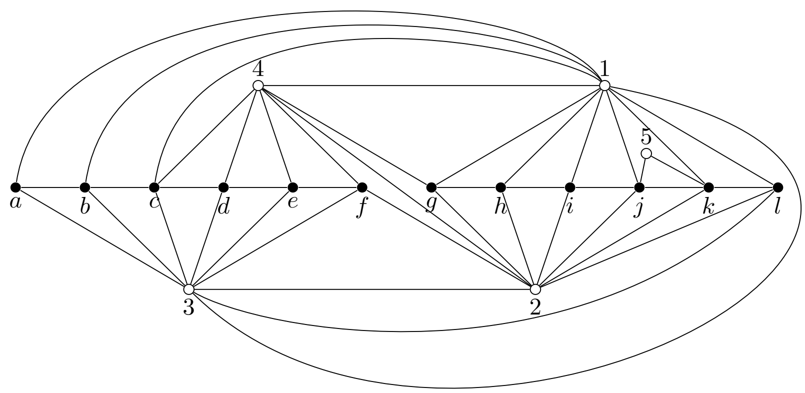


FIGURE 4. A planar decomposition of $K_5 + P_{12}$

Property 5.2. *Let G and H be two simple graphs with order p and q respectively, then*

$$t(G + H) \leq Min\{t(H) + t(G + \overline{K}_q), t(G) + t(H + \overline{K}_p)\}.$$

Proof. The graph $G + H$ is a subgraph of $H \cup (G + \overline{K}_q)$ and it is also a subgraph of $G \cup (H + \overline{K}_p)$, in which $V(\overline{K}_q) = V(H)$ and $V(\overline{K}_p) = V(G)$. Hence both $t(G+H) \leq t(H)+t(G+\overline{K}_q)$ and $t(G+H) \leq t(G)+t(H+\overline{K}_p)$ hold, the property follows. $\square$

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