DEPENDENCE OF EIGENVALUES ON THE DIFFUSION OPERATORS WITH RANDOM JUMPS FROM THE BOUNDARY

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ABSTRACT. This paper deals with a non-self-adjoint eigenvalue problem

$$\begin{cases} a(x)y''(x) + b(x)y'(x) = \lambda y(x), \\ y(0) = \int_0^1 y(x)\mathrm{d}\nu_0(x),\ y(1) = \int_0^1 y(x)\mathrm{d}\nu_1(x), \end{cases}$$

which is associated with the generator of one dimensional diffusions with random jumps from the boundary. We focus on the dependence of spectral gap, eigenvalues and eigenfunctions on the coefficients a, b and the probability distributions ν_0, ν_1. To prove this, we show that all the eigenvalues are confined to a parabolic neighborhood of the real axis. Moreover, we also prove that zero is an algebraically simple eigenvalue of the problem.

1. INTRODUCTION

In this paper, we consider the following non-self-adjoint eigenvalue problem

$$\begin{cases} ly(x) := a(x)y''(x) + b(x)y'(x) = \lambda y(x),\ x \in (0,1)\,, \\ y(0) = \int_0^1 y(x)\mathrm{d}\nu_0(x),\ y(1) = \int_0^1 y(x)\mathrm{d}\nu_1(x), \end{cases} \tag{1.1}$$

where

$$\nu_0 \text{ and } \nu_1 \text{ are probability distributions on } (0,1) \tag{1.2}$$

and

$$a \in W^{2,2}(J,\mathbb{R}),\ b \in W^{1,2}(J,\mathbb{R}),\ a < 0 \text{ on } J = [0,1]. \tag{1.3}$$

It is well known that the differential operator associated with the problem (1.1) is the generator of one dimensional diffusions with random jumps from the boundary.

In the last decades, diffusions with random jumps from the boundary have attracted enormous interest for various probability considerations and practical interests in genetics (see, e.g., [1–11] and the references therein). These start with the fundamental work of W. Feller ([7, 12]) which characterized completely the analytic structure of one-dimensional diffusion processes and which referred to such a process as "instantaneous return process". The process itself can be easily described. Consider a diffusion process with initial value $x \in D$ in an open domain $D \subset \mathbb{R}^d$ which we assume to have smooth boundary. Let $\{\nu_\xi : \xi \in \partial D\}$ be a family of probability distributions on the domain D. When the boundary point $\xi \in \partial D$ is reached, an instantaneous return into the interior is effected according to the

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probability distribution ν_ξ and the process starts afresh. The same mechanism is repeated independently each time the process reaches the boundary. Such a process is ergodic and its distribution converges in total variation exponentially fast to its invariant measure. What should be mentioned here is that in [2, Theorem 1], I. Ben-Ari and R. G. Pinsky provided a characterization of the rate in terms of the spectral gap/eigenvalue problem corresponding to the generator of the process.

Thus it is fair to say that spectral analysis of the generator, which gave rise to several interesting results recently, have probabilistic significance (see, e.g., [1, 6, 9, 10, 13]). In this context, it is then important to understand how a change of the generator affects the spectral gap, the eigenvalues and eigenfunctions. When the probability distribution ν_ξ is independent of the point of exit ξ, M. Kolb and A. Wubker established in [10] the continuous dependence of the spectral gap on ν_ξ, which answers a question posed by I. Ben-Ari and R. Pinsky in [1]. Here the continuity is meant with respect to the weak topology. This is our starting point and we aim to further discuss the dependence of the eigenvalues, the spectral gap and eigenfunctions on all the parameters of the problem (1.1) including the diffusion coefficient a, the drift coefficient b and the probability distributions ν_0, ν_1. To the best of our knowledge, such questions have not been investigated before.

Let us now briefly present the results of this paper. In Section 2, to undertake our study on the problem (1.1) , we first introduce a "boundary value problem space" $\Omega = \{\omega = (a, b, \nu_0, \nu_1); \text{(1.2) and (1.3) hold}\}$. Here each element $\omega = (a, b, \nu_0, \nu_1) \in \Omega$ represents an eigenvalue problem (1.1). By an eigenvalue of $\omega \in \Omega$ we mean an eigenvalue of the problem (1.1). For the topology of Ω we use a metric d defined as follows: for $\omega = (a, b, \nu_0, \nu_1) \in \Omega$, $\omega_0 = (a_0, b_0, \nu_0^0, \nu_1^0) \in \Omega$, define

$$d(\omega, \omega_0) = \int_0^1 \left(\left| \frac{1}{a} - \frac{1}{a_0} \right| + \left| \frac{b}{a} - \frac{b_0}{a_0} \right| + \left| \nu_0 - \nu_0^0 \right| + \left| \nu_1 - \nu_1^0 \right| \right). \tag{1.4}$$

It can be shown in this section that each eigenvalue of the problem (1.1) can be embedded into a continuous eigenvalue branch (see Theorem 2.11). More precisely, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\omega = (a, b, \nu_0, \nu_1) \in \Omega$ satisfies

$$d(\omega, \omega_0) < \delta,$$

then the problem ω has an eigenvalue $\lambda(\omega)$ satisfying

$$|\lambda(\omega) - \lambda(\omega_0)| < \epsilon, \tag{1.5}$$

where $\lambda(\omega_0)$ is assumed to be an eigenvalue of the problem $\omega_0 = (a_0, b_0, \nu_0^0, \nu_1^0) \in \Omega$. In view of this, eigenfunctions can be found which depend continuously on the eigenvalue problem in the uniform norm (see Proposition 2.16).

Section 3 provides a characterization of the existence region of the eigenvalues, that is, there exists a positive constant R_0 such that all the eigenvalues of the problem (1.1) lie in the region

$$\Lambda = \left\{ \lambda \in \mathbb{C} \,\middle|\, \mathrm{Re}\lambda > \frac{(\mathrm{Im}\lambda)^2}{4R_0^2} - R_0^2 \right\}.$$

This leads us to give a definition (Remark 3.2) of the m-th eigenvalue λ_m, $m \in \mathbb{N}_0$. Note that in this paper $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ denotes the set of positive integers.

Based on the meaning of λ_m, a natural question arises by noticing that there is no index on the eigenvalue in (1.5): what can be said about the dependence of the

m-th eigenvalue λ_m on the eigenvalue problem (1.1)? This is the main question we want to address in Section 4. Let us now present our principal result.

Theorem 1.1. *Let $\omega = (a, b, \nu_0, \nu_1) \in \Omega$ and $\omega_n = (a_n, b_n, \nu_{0,n}, \nu_{1,n}) \in \Omega$, $n \in \mathbb{N}$. Assume that*

$$\|a_n - a\|_{W^{2,2}} \to 0, \|b_n - b\|_{W^{1,2}} \to 0, \nu_{0,n} \xrightarrow{w} \nu_0, \nu_{1,n} \xrightarrow{w} \nu_1, \text{ as } n \to \infty. \tag{1.6}$$

Denote the m-th eigenvalue of ω_n by $\lambda_m(\omega_n)$.

(1) *Denote the eigenvalues of ω as follows,*

$$\lambda_0^0(\omega), \lambda_1^0(\omega), \cdots, \lambda_{k_1-1}^0(\omega); \lambda_{k_1}^1(\omega), \lambda_{k_1+1}^1(\omega), \cdots, \lambda_{k_2-1}^1(\omega); \cdots ;$$
$$\lambda_{k_j}^j(\omega), \lambda_{k_j+1}^j(\omega), \cdots, \lambda_{k_{j+1}-1}^j(\omega); \cdots$$

where

$$Re\lambda_{k_j}^j(\omega) = Re\lambda_{k_j+1}^j(\omega) = \cdots = Re\lambda_{k_{j+1}-1}^j(\omega) < Re\lambda_{k_{j+1}}^{j+1}(\omega),$$
$$Im\lambda_{k_j}^j(\omega) \le Im\lambda_{k_j+1}^j(\omega) \le \cdots \le Im\lambda_{k_{j+1}-1}^j(\omega), \ j \in \mathbb{N}_0, \ k_j \in \mathbb{N}_0,$$
$$k_0 = 0 \text{ and } k_0 < k_1 < \cdots < k_j < \cdots .$$

Then for each $j \in \mathbb{N}_0$, given any $\epsilon > 0$, there exists a number $N > 0$ such that if $n > N$, one has

$$\sum_{m=k_j}^{k_{j+1}-1} \left|\lambda_{m_n}(\omega_n) - \lambda_m^j(\omega)\right| < \epsilon,$$

where the index set $\{m_n : m = k_j, \ldots, k_{j+1}-1\} = \{m : m = k_j, \ldots, k_{j+1}-1\}$;

(2) *For brevity denote the m-th eigenvalue of ω by $\lambda_m(\omega)$. Then for each $m \in \mathbb{N}_0$, one has $Re\lambda_m(\omega_n) \to Re\lambda_m(\omega)$ as $n \to \infty$.*

Theorem 1.1 gives the dependence of the m-th eigenvalue λ_m on the problem (1.1) and illustrates the continuous dependence of $\mathrm{Re}\lambda_m$ on the coefficients a and b with respect to the topologies induced by $\|\cdot\|_{W^{2,2}}$ and $\|\cdot\|_{W^{1,2}}$, respectively. In addition, Theorem 1.1 shows that $\mathrm{Re}\lambda_m$ depends continuously on the distributions ν_0, ν_1 with respect to the weak topology. It should be mentioned that in [17], Q. Kong, H. Wu and A. Zettl discussed the dependence of the m-th eigenvalue on the classical self-adjoint Sturm-Liouville problems; they proved the continuous dependence of the m-th eigenvalue on the coefficients of the Sturm-Liouville equation with respect to the topology induced by the norm $\|\cdot\|_{L^1}$ and completely characterized the discontinuity of the m-th eigenvalue as a functional on the space $B_S^{\mathbb{C}}$ of self-adjoint boundary conditions [17, Theorem 2.1, Lemma 3.32, Theorem 3.39]. It seems to the authors that the approach used in [17] cannot be adopted to the non-self-adjoint problem (1.1) in a direct way, since even though the differential equation in (1.1) is symmetric (see Remark 2.4), there still exists the possibility of complex eigenvalues due to the non-local boundary conditions. Thus many new ideas and additional effort are required. To prove Theorem 1.1, we consider a stronger topology than that induced by (1.4), and then prove that all the eigenvalues of ω_n and ω are confined to a parabolic neighborhood of the real axis (Lemma 4.1) under the assumption (1.6), which is the key to establish our dependence results.

Finally, as a consequence of Theorem 1.1, Section 5 is devoted to prove the continuity dependence of the spectral gap on the coefficients a, b and distributions ν_0, ν_1 (see Theorem 1.2). We start this section by noticing that zero is an algebraically simple eigenvalue of the problem (1.1) and all the nonzero eigenvalues have strictly positive real part (see Proposition 5.1).

Theorem 1.2. *Let* $\omega = (a, b, \nu_0, \nu_1) \in \Omega$ *and* $\omega_n = (a_n, b_n, \nu_{0,n}, \nu_{1,n}) \in \Omega$, $n \in \mathbb{N}$. *If* $\|a_n - a\|_{W^{2,2}} \to 0, \|b_n - b\|_{W^{1,2}} \to 0$, $\nu_{0,n} \overset{w}{\to} \nu_0, \nu_{1,n} \overset{w}{\to} \nu_1$, *as* $n \to \infty$, *one has* $\gamma_1(\omega_n) \to \gamma_1(\omega)$, *as* $n \to \infty$, *where*

$$\gamma_1(\omega) := \inf \{ Re\lambda | \, \lambda \textit{ is an eigenvalue of the problem } \omega \textit{ and } \lambda \neq 0 \}$$

and

$$\gamma_1(\omega_n) := \inf \{ Re\lambda | \, \lambda \textit{ is an eigenvalue of the problem } \omega_n \textit{ and } \lambda \neq 0 \} .$$

Despite our purely theoretical study we want to mention that the dependence results considered in this paper might have some importance on the numerical computation of the spectral gap, eigenvalues and eigenfunctions of the problem (1.1).

2. Continuity of Eigenvalues and Eigenfunctions

In this section, we show that the eigenvalues and eigenfunctions depend continuously on the problem (1.1), i.e., a "small" change of the problem results in a "small" change of each eigenvalue and each eigenfunction (see Theorem 2.11 and Proposition 2.16). Let us first state several preliminary facts.

Lemma 2.1. *The initial value problem consisting of the differential equation in* (1.1) *and the initial conditions*

$$y(0) = h, \; y'(0) = k, \; h, \; k \in \mathbb{C} \tag{2.1}$$

has a unique solution $y(x, \lambda)$. *And each of the functions* $y(x, \lambda)$ *and* $y'(x, \lambda)$ *is continuous on* $[0, 1] \times \mathbb{C}$. *In particular, the functions* $y(x, \lambda)$ *and* $y'(x, \lambda)$ *are entire functions of* $\lambda \in \mathbb{C}$.

Proof. See [15]. □

Remark 2.2. In fact, it follows from [15] that the derivative of $y(x, \lambda)$ with respect to λ is given by

$$y'_\lambda(x, \lambda) = \int_0^x \frac{y_2(x, \lambda) y_1(t, \lambda) - y_1(x, \lambda) y_2(t, \lambda)}{a(t) \exp\left(-\int_0^t \frac{b(s)}{a(s)} \mathrm{d}s\right)} y(t, \lambda) \mathrm{d}t.$$

Remark 2.3. In this section, without considering the existence of eigenvalues, assumptions

$$\frac{1}{a}, \; \frac{b}{a} \in L^1(J, \mathbb{C})$$

are sufficient for all the statements to be valid. However, in Section 3-5, the assumptions (1.3) are still needed.

Remark 2.4. Define $c(x) := \exp\left(\int_0^x \frac{b(t)}{a(t)} dt\right)$ and $w(x) := -\frac{c(x)}{a(x)}$. Then the differential equation in (1.1) can be rewritten as the following form:

$$-\left(c(x) y'(x)\right)' = \lambda w(x) y(x), \; x \in (0, 1) .$$

Lemma 2.5. *Consider the initial value problem consisting of the differential equation in* (1.1) *and the initial conditions*

$$y(0) = h,\ y'(0) = k,\ h,\ k \in \mathbb{C}. \tag{2.2}$$

Denote the unique solution by $y\left(\cdot, h, k, a, b, \lambda\right).$ *Then, for any given* $\epsilon > 0$, *there exists a number* $\delta > 0$ *such that if*

$$|\lambda - \lambda_0| + |h - h_0| + |k - k_0| + \int_0^1 \left(\left| \frac{1}{a} - \frac{1}{a_0} \right| + \left| \frac{b}{a} - \frac{b_0}{a_0} \right| \right) < \delta,$$

then

$$|y(x, h, k, a, b, \lambda) - y(x, h_0, k_0, a_0, b_0, \lambda_0)| < \epsilon$$

and

$$|y'(x, h, k, a, b, \lambda) - y'(x, h_0, k_0, a_0, b_0, \lambda_0)| < \epsilon$$

uniformly for all $x \in [0, 1]$.

Proof. This is a consequence of [15, Theorem 1.6.2]. □

Let y_1 and y_2 be the fundamental solutions of the differential equation in (1.1) determined by the initial conditions

$$y_1(0, \lambda) = y_2'(0, \lambda) = 1,\ y_2(0, \lambda) = y_1'(0, \lambda) = 0,\ \lambda \in \mathbb{C}. \tag{2.3}$$

Lemma 2.6. *A number* λ *is an eigenvalue of the problem* (1.1) *if and only if*

$$\Delta(\lambda) := \det \begin{pmatrix} \int_0^1 y_1(x, \lambda) \mathrm{d}\nu_0(x) - 1 & \int_0^1 y_2(x, \lambda) \mathrm{d}\nu_0(x) \\ \int_0^1 y_1(x, \lambda) \mathrm{d}\nu_1(x) - y_1(1, \lambda) & \int_0^1 y_2(x, \lambda) \mathrm{d}\nu_1(x) - y_2(1, \lambda) \end{pmatrix} = 0. \tag{2.4}$$

Remark 2.7. Note that in this paper the **algebraic multiplicity** of an eigenvalue is its order as a zero of the characteristic function $\Delta(\lambda)$.

Remark 2.8. If $a(x)$, $b(x)$ are all real-valued, one has $y_i(x, \overline{\lambda}) = \overline{y_i(x, \lambda)}$, $\lambda \in \mathbb{C}$, $i = 1, 2$. Thus $\Delta(\overline{\lambda}) = \overline{\Delta(\lambda)}$. This implies that if λ_* is an eigenvalue of the problem (1.1), then so is $\overline{\lambda_*}$. Moreover, the algebraic multiplicity of λ_* and $\overline{\lambda_*}$ are equal. (Note that in this paper the overbar means the complex conjugate.)

Remark 2.9. In this paper, the assumption (1.2) means that ν_i, $i = 0, 1$, are distribution functions (non-decreasing functions) with $\lim\limits_{x \to 0^+} \nu_i(x) = \nu_i(0) = 0$ and $\lim\limits_{x \to 1^-} \nu_i(x) = \nu_i(1) = 1$. And we refer to [7] for the detailed probabilistic background of the diffusion process associated with (1.1).

Lemma 2.10. (*Continuity of the zeros of an analytic function*). *Let* A *be an open set in the complex plane* $\mathbb{C}$, F *a metric space,* f *a continuous complex valued function on* $A \times F$ *such that for each* $\alpha \in F$, *the map* $z \to f(z, \alpha)$ *is an analytic function on* A. *Let* B *be an open subset of* A *whose closure* $\overline{B}$ *in* $\mathbb{C}$ *is compact and contained in* A, *and let* $\alpha_0 \in F$ *be such that no zero of* $f(z, \alpha_0)$ *is on the boundary of* B. *Then there exists a neighborhood* W *of* α_0 *in* F *such that* :

(*a*) *For any* $\alpha \in W$, $f(z, \alpha)$ *has no zero on the boundary of* B.

(*b*) *For any* $\alpha \in W$, *the sum of the orders of the zeros of* $f(z, \alpha)$ *contained in* B *is independent of* α.

Proof. See 9.17.4 in [16]. □

Now we aim to show that the eigenvalues are continuous functions of all the parameters of the problem including the coefficients a, b and distributions ν_0, ν_1 (Theorem 2.11). Recall the "boundary value problem space" $\Omega = \{\omega = (a, b, \nu_0, \nu_1);$ (1.2) and (1.3) hold$\}$ defined in the introduction.

Theorem 2.11. *Let $\omega_0 = (a_0, b_0, \nu_0^0, \nu_1^0) \in \Omega$. Assume that $\lambda(\omega_0)$ is an eigenvalue of the problem ω_0. Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $\omega = (a, b, \nu_0, \nu_1) \in \Omega$ satisfies*

$$d(\omega, \omega_0) < \delta, \tag{2.5}$$

then the problem ω has an eigenvalue $\lambda(\omega)$ satisfying

$$|\lambda(\omega) - \lambda(\omega_0)| < \epsilon. \tag{2.6}$$

Proof. For any problem $\omega = (a, b, \nu_0, \nu_1) \in \Omega$, denote the characteristic function introduced in Lemma 2.6 by

$$\begin{aligned} &\Delta(\omega, \lambda) \\ =& \det \begin{pmatrix} \int_0^1 y_1(x, a, b, \lambda) \mathrm{d}\nu_0 - 1 & \int_0^1 y_2(x, a, b, \lambda) \mathrm{d}\nu_0 \\ \int_0^1 y_1(x, a, b, \lambda) \mathrm{d}\nu_1 - y_1(1, a, b, \lambda) & \int_0^1 y_2(x, a, b, \lambda) \mathrm{d}\nu_1 - y_2(1, a, b, \lambda) \end{pmatrix}. \end{aligned}$$

We first show that $\Delta(\omega, \lambda)$ is an entire function of $\lambda \in \mathbb{C}$ and is continuous in $\omega \in \Omega$. In fact, the analyticity of $\Delta(\omega, \lambda)$ on λ follows directly from Lemma 2.1. Moreover, integrating by parts shows that for $\omega = (a, b, \nu_0, \nu_1) \in \Omega$ and $\omega_0 = (a_0, b_0, \nu_0^0, \nu_1^0) \in \Omega$,

$$\begin{aligned} & \left| \int_0^1 y_2(x, a, b, \lambda) \mathrm{d}\nu_1(x) - \int_0^1 y_2(x, a_0, b_0, \lambda) \mathrm{d}\nu_1^0(x) \right| \\ \leq \; & |y_2(1, a, b, \lambda) - y_2(1, a_0, b_0, \lambda)| \\ & + \left| \int_0^1 y_2'(x, a, b, \lambda) \nu_1(x) \mathrm{d}x - \int_0^1 y_2'(x, a_0, b_0, \lambda) \nu_1(x) \mathrm{d}x \right| \\ & + \left| \int_0^1 y_2'(x, a_0, b_0, \lambda) \nu_1(x) \mathrm{d}x - \int_0^1 y_2'(x, a_0, b_0, \lambda) \nu_1^0(x) \mathrm{d}x \right|. \end{aligned}$$

Hence we conclude from Lemma 2.5 that $\int_0^1 y_2(x, a, b, \lambda) \mathrm{d}\nu_1(x)$ is continuous in $\omega \in \Omega$. An analogous proof yields that $\Delta(\omega, \lambda)$ is continuous in $\omega \in \Omega$.

From Lemma 2.6, we know that $\lambda(\omega)$ is an eigenvalue if and only if

$$\Delta(\omega, \lambda(\omega)) = 0.$$

This means that if μ is an eigenvalue of the problem ω_0, then $\Delta(\omega_0, \mu) = 0$. Since μ is an isolated eigenvalue of the problem ω_0, there exists $\eta > 0$, such that $\Delta(\omega_0, \lambda) \neq 0$ for $\lambda \in \Gamma_\eta := \{\lambda \in \mathbb{C} \,||\lambda - \mu| = \eta\}$. Thus the statement of Proposition (2.11) follows from Lemma 2.10. □

Remark 2.12. In this paper, the *algebraic multiplicity* of an eigenvalue is the order of it as a zero of the characteristic function $\Delta(\lambda)$ defined in Lemma 2.6. The linear space spanned by the eigenfunctions for an eigenvalue is called the *eigenspace* for the eigenvalue. The *geometric multiplicity* of an eigenvalue is defined to be the dimension of its eigenspace, which is either 1 or 2.

Remark 2.13. If the algebraic multiplicity of $\lambda(\omega_0)$ is χ, it follows from Theorem 2.11 and Lemma 2.6 that for ω satisfying (2.5) , (2.6) holds for χ eigenvalues of ω. In other words, each eigenvalue of algebraic multiplicity χ is on χ locally continuous eigenvalue branches. Multiple eigenvalues are counted according to their algebraic multiplicity.

Remark 2.14. Let $\omega_0 = (a_0, b_0, \nu_0^0, \nu_1^0) \in \Omega$. Assume that Γ is any contour such that ω_0 has no eigenvalue on it and m eigenvalues inside it. Then there exists a neighborhood U of ω_0 in Ω such that any $\omega \in U$ also has exactly m eigenvalues inside the contour Γ. Here eigenvalues are counted according to their algebraic multiplicity.

Remark 2.15. Note that there is no index on λ in (2.6) , thus it is natural to pay attention to the dependence of the m-th eigenvalue λ_m on the problem. This will be addressed in Section 4 and the meaning of λ_m will be given in Remark 3.2.

To conclude this section, we prove the following proposition, which illustrates the dependence of eigenfunctions on the problem (1.1).

Proposition 2.16. Let $\omega_0 = (a_0, b_0, \nu_0^0, \nu_1^0) \in \Omega$. Assume $\lambda(\omega)$ be the continuous eigenvalue branch through the eigenvalue $\lambda(\omega_0)$. Then the following statements are valid.

(i) Assume the eigenvalue $\lambda(\omega_0)$ is geometrically simple and let $u = u(\cdot, \omega_0)$ denote an eigenfunction of the eigenvalue $\lambda(\omega_0)$. Then there exists a neighborhood $M \subset \Omega$ of ω_0 such that $\lambda(\omega)$ is simple for every ω in M. Moreover, there exist eigenfunctions $u = u(\cdot, \omega)$ of $\lambda(\omega)$ such that

$$u(\cdot, \omega) \to u(\cdot, \omega_0),\ u'(\cdot, \omega) \to u'(\cdot, \omega_0),\ \text{as } \omega \to \omega_0 \text{ in } \Omega, \tag{2.7}$$

both uniformly on the interval $[0, 1]$.

(ii) Assume that $\lambda(\omega)$ is a geometrically double eigenvalue for all ω in some neighborhood $N \subset \Omega$ of ω_0. Let $u = u(\cdot, \omega_0)$ be any eigenfunction of the eigenvalue $\lambda(\omega_0)$. Then there exist eigenfunctions $u = u(\cdot, \omega)$ of $\lambda(\omega)$ such that

$$u(\cdot, \omega) \to u(\cdot, \omega_0),\ u'(\cdot, \omega) \to u'(\cdot, \omega_0),\ \text{as } \omega \to \omega_0 \text{ in } \Omega, \tag{2.8}$$

both uniformly on the interval $[0, 1]$.

Proof. (i) Denote $\omega := (a, b, \nu_0, \nu_1)$ and $D(\omega) :=$
$$\begin{pmatrix} \int_0^1 y_1(x, a, b, \lambda(\omega))\mathrm{d}\nu_0 - 1 & \int_0^1 y_2(x, a, b, \lambda(\omega))\mathrm{d}\nu_0 \\ \int_0^1 y_1(x, a, b, \lambda(\omega))\mathrm{d}\nu_1 - y_1(1, a, b, \lambda(\omega)) & \int_0^1 y_2(x, a, b, \lambda(\omega))\mathrm{d}\nu_1 - y_2(1, a, b, \lambda(\omega)) \end{pmatrix}.$$
Lemma 2.6 implies that $\det D(\omega) = 0$ and thus $0 \leq \mathrm{rank} D(\omega_0) < 2$. Since $\lambda(\omega_0)$ is geometrically simple, we have $\mathrm{rank} D(\omega_0) = 1$. Hence without loss of generality, assume that

$$\int_0^1 y_1(x, a_0, b_0, \lambda(\omega_0))\mathrm{d}\nu_0^0(x) - 1 \neq 0.$$

From the proof of Theorem 2.11, it follows that $\int_0^1 y_1(x, a, b, \lambda)\mathrm{d}\nu_0(x)$ is continuous in $\omega \in \Omega$. Thus there exists a neighborhood M of ω_0 such that for every $\omega = (a, b, \nu_0, \nu_1) \in M$,

$$\int_0^1 y_1(x, a, b, \lambda(\omega))\mathrm{d}\nu_0(x) - 1 \neq 0,$$

and hence rank$D(\omega)=1$. For each $\omega\in M$, denote

$$\left(\begin{array}{c} c_1(\omega) \\ c_2(\omega) \end{array}\right) := \left(\begin{array}{c} -\int_0^1 y_2(x,a,b,\lambda(\omega))\mathrm{d}\nu_0\cdot\left(\int_0^1 y_1(x,a,b,\lambda(\omega))\mathrm{d}\nu_0-1\right)^{-1} \\ 1 \end{array}\right).$$

Then direct calculation yields $D(\omega)\left(\begin{array}{c} c_1(\omega) \\ c_2(\omega) \end{array}\right)=\left(\begin{array}{c} 0 \\ 0 \end{array}\right)$. This implies that for each $\omega\in M$,

$$c_1(\omega)y_1(x,a,b,\lambda(\omega))+c_2(\omega)y_2(x,a,b,\lambda(\omega))$$

is an eigenfunction corresponding to $\lambda(\omega)$. Moreover, it is obvious that $\left(\begin{array}{c} c_1(\omega) \\ c_2(\omega) \end{array}\right)$ is continuous in ω_0. This together with Lemma 2.5 directly yield (2.7) .

(ii) Note that $\lambda(\omega)$ is a geometrically double eigenvalue for all ω in some neighborhood N of ω_0 in Ω. Thus we can choose eigenfunctions $u=u(\cdot,\omega)$ of $\lambda(\omega)$ all of which satisfy the same initial condition at 0 since a linear combination of two linear independent eigenfunctions can be chosen to satisfy initial conditions. Then (2.8) follows from Lemma 2.5. $\square$

3. The existence region of eigenvalues

Since the eigenvalue problem (1.1) is non-self-adjoint, there exists the possibility of complex eigenvalues. Hence in this section we focus on the existence region of the complex eigenvalues (see Theorem 3.1), which plays a key role in analyzing the dependence of the m-th eigenvalue λ_m on the eigenvalue problem.

Theorem 3.1. *Given any problem* (1.1) , *there exists a positive constant R_0 such that all its eigenvalues lie in the region*

$$\Lambda=\left\{\lambda\in\mathbb{C}\,\middle|\, Re\lambda>\frac{(Im\lambda)^2}{4R_0^2}-R_0^2\right\}.$$

Remark 3.2. On the basis of the above statement, we denote by λ_m, $m\in\mathbb{N}_0$, the eigenvalues of the problem (1.1) counted with algebraic multiplicities and arranged by increasing of their real parts; if real parts equal, then we arrange the eigenvalues by increasing of their imaginary parts. In other words, the eigenvalues λ_m, $m\in\mathbb{N}_0$, are arranged such that $\mathrm{Re}\lambda_0\le\mathrm{Re}\lambda_1\le\mathrm{Re}\lambda_2\le\cdots\le\mathrm{Re}\lambda_m\le\cdots$, with the additional condition that $\mathrm{Im}\lambda_m\le\mathrm{Im}\lambda_{m+1}$ whenever $\mathrm{Re}\lambda_m=\mathrm{Re}\lambda_{m+1}$.

Before giving the proof of Theorem 3.1, we first introduce some preliminary facts. Denote $l:=\int_0^1\frac{\mathrm{d}s}{\sqrt{-a(s)}}$ and let $t=t(x)=l^{-1}\cdot\int_0^x\frac{\mathrm{d}s}{\sqrt{-a(s)}}$, then the problem (1.1) becomes

$$\left\{\begin{array}{l} -y''(t)+p(t)y'(t)=\lambda l^2y(t), t\in(0,1), \\ y(0)=\int_0^1 y(t)\mathrm{d}\widetilde{\nu}_0(t),\ y(1)=\int_0^1 y(t)\mathrm{d}\widetilde{\nu}_1(t), \end{array}\right. \tag{3.1}$$

where $p(t)=\frac{l\left(b(x)-\frac{1}{2}a'(x)\right)}{\sqrt{-a(x)}}$, $\widetilde{\nu}_i(t)=\nu_i(x)$, $i=0,1$. Under the transformation $v(t)=\exp\left(\frac{-\int_0^t p(s)\mathrm{d}s}{2}\right)y(t)$, problem (3.1) is equivalent to the following eigenvalue problem

$$\left\{\begin{array}{l} -v''(t)+q(t)v(t)=\lambda l^2v(t), t\in(0,1), \\ v(0)=\int_0^1 P(t)v(t)\mathrm{d}\widetilde{\nu}_0(t),\ P(1)v(1)=\int_0^1 P(t)v(t)\mathrm{d}\widetilde{\nu}_1(t), \end{array}\right. \tag{3.2}$$

where $q(t) = \frac{1}{4}p^2(t) - \frac{1}{2}p'(t)$, $P(t) = \exp\left(\frac{\int_0^t p(s)\mathrm{d}s}{2}\right)$. Note that the assumption (1.3) clearly implies that $p \in W^{1,2}(J,\mathbb{R})$ and $q \in L^1(J,\mathbb{R})$. Let $v_1(t,\lambda)$ and $v_2(t,\lambda)$ be the fundamental solutions of the differential equation in (3.2) determined by the initial conditions

$$v_1(0,\lambda) = v_2'(0,\lambda) = 1,\ v_2(0,\lambda) = v_1'(0,\lambda) = 0,\ \lambda \in \mathbb{C}.$$

Denote

$$\Delta_1(\lambda) :=$$
$$\det\begin{pmatrix} \int_0^1 P(t)v_1(t,\lambda)\mathrm{d}\widetilde{\nu}_0(t) - 1 & \int_0^1 P(t)v_2(t,\lambda)\mathrm{d}\widetilde{\nu}_0(t) \\ \int_0^1 P(t)v_1(t,\lambda)\mathrm{d}\widetilde{\nu}_1(t) - P(1)v_1(1,\lambda) & \int_0^1 P(t)v_2(t,\lambda)\mathrm{d}\widetilde{\nu}_1(t) - P(1)v_2(1,\lambda) \end{pmatrix}.$$

Lemma 3.3. *A number λ is an eigenvalue of the eigenvalue problem* (3.2) *if and only if* $\Delta_1(\lambda) = 0$.

Proof. This result follows from a direct calculation. □

Remark 3.4. A number λ is an eigenvalue of the problem (3.2) if and only if it is an eigenvalue of the problem (1.1). Moreover, it follows from a direct calculation that $\Delta_1(\lambda) \equiv \Delta(\lambda)$. Recall that $\Delta(\lambda)$ is the characteristic function defined in (2.4).

Based on the above statements, to prove Theorem 3.1, it is sufficient to analyze the existence region of zeros of $\Delta_1(\lambda)$. Now we first give an estimate on $\Delta_1(\lambda)$, which plays an important role in what follows. Let us use $\mathrm{Im}\sqrt{\lambda}$ to denote the imaginary part of $\sqrt{\lambda}$, where the argument of the square-root function is chosen so that $\arg(\sqrt{\lambda}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For brevity we will often use the notation $\|q\|_1 := \|q\|_{L^1(J,\mathbb{R})}$.

Lemma 3.5. *The characteristic function $\Delta_1(\lambda)$ satisfies*

$$|\Delta_1(\lambda) - \Delta_0(\lambda)| \le C\frac{1}{l^2|\lambda|}\exp\left(l\left|Im\sqrt{\lambda}\right| + \|q\|_1\right) \tag{3.3}$$

where $C = 4\exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)$ *and*

$$\begin{aligned}\Delta_0(\lambda) :&= \int_0^1\int_0^1 P(s)P(t)\frac{\sin\left(l\sqrt{\lambda}\,(s-t)\right)}{l\sqrt{\lambda}}\mathrm{d}\widetilde{\nu}_0(t)\mathrm{d}\widetilde{\nu}_1(s) \\ &\quad -P(1)\int_0^1 P(t)\frac{\sin\left(l\sqrt{\lambda}\,(1-t)\right)}{l\sqrt{\lambda}}\mathrm{d}\widetilde{\nu}_0(t) \\ &\quad -\int_0^1 P(t)\frac{\sin\left(l\sqrt{\lambda}t\right)}{l\sqrt{\lambda}}\mathrm{d}\widetilde{\nu}_1(t) + P(1)\frac{\sin\left(l\sqrt{\lambda}\right)}{l\sqrt{\lambda}}.\end{aligned}$$

Proof. In order to give the estimation, now we define a function

$$\Psi(s,t,\lambda) := v_2(s,\lambda)v_1(t,\lambda) - v_1(s,\lambda)v_2(t,\lambda).$$

Hence from the definition of $\Delta_1(\lambda)$, it follows that

$$\begin{aligned}
(3.4)\qquad & \Delta_1(\lambda)-\Delta_0(\lambda)\\
= & \int_0^1\int_0^1 P(s)P(t)\left[\Psi(s,t,\lambda)-\frac{\sin\left(l\sqrt{\lambda}\,(s-t)\right)}{l\sqrt{\lambda}}\right]\mathrm{d}\widetilde{\nu}_0(t)\mathrm{d}\widetilde{\nu}_1(s)\\
& -P(1)\int_0^1 P(t)\left[\Psi(1,t,\lambda)-\frac{\sin\left(l\sqrt{\lambda}\,(1-t)\right)}{l\sqrt{\lambda}}\right]\mathrm{d}\widetilde{\nu}_0(t)\\
& -\int_0^1 P(t)\left[v_2(t,\lambda)-\frac{\sin(l\sqrt{\lambda}t)}{l\sqrt{\lambda}}\right]\mathrm{d}\widetilde{\nu}_1(t)+P(1)\left[v_2(1,\lambda)-\frac{\sin(l\sqrt{\lambda})}{l\sqrt{\lambda}}\right].
\end{aligned}$$

In order to prove (3.3), we first show that for $(s,t)\in[0,1]\times[0,1]$ and $\lambda\in\mathbb{C}\backslash\{0\}$, the following estimate

$$(3.5)\qquad \left|\Psi(s,t,\lambda)-\frac{\sin(l\sqrt{\lambda}\,(s-t))}{l\sqrt{\lambda}}\right|\le\frac{1}{l^2\,|\lambda|}\exp\left(\left|l\mathrm{Im}\sqrt{\lambda}\right||s-t|+\|q\|_1\right)$$

is valid. In fact, for fixed $t\in[0,1]$, as a function of s, $\Psi(s,t,\lambda)$ is a solution of the equation

$$-v''(s)+q(s)v(s)=\lambda l^2 v(s)$$

determined by the initial conditions

$$\begin{aligned}
(3.6)\qquad & \Psi(s,t,\lambda)|_{s=t} &=& \ 0,\\
(3.7)\qquad & \left.\frac{\mathrm{d}\Psi(s,t,\lambda)}{\mathrm{d}s}\right|_{s=t} &=& \ \det\begin{pmatrix} v_1(t,\lambda) & v_2(t,\lambda)\\ v_1'(t,\lambda) & v_2'(t,\lambda)\end{pmatrix}=1,\ \lambda\in\mathbb{C}.
\end{aligned}$$

Hence analogous to [18], for $(s,t,\lambda)\in[0,1]\times[0,1]\times\mathbb{C}$, $\Psi(s,t,\lambda)$ can be written as the following form:

$$(3.8)\qquad \Psi(s,t,\lambda)=\frac{\sin(l\sqrt{\lambda}\,(s-t))}{l\sqrt{\lambda}}+\sum_{n\ge1}S_n(s,t,\lambda)$$

where

$$\begin{aligned}
& S_n(s,t,\lambda)\\
= & \begin{cases}\int_{0\le t_1\le\cdots\le t_{n+1}:=s-t} s_\lambda(t_1)\prod_{i=1}^{n} s_\lambda(t_{i+1}-t_i\,)q(t_i+t)\mathrm{d}t_1\cdots\mathrm{d}t_n, t<s,\\ -\int_{0\le t_1\le\cdots\le t_{n+1}:=t-s} s_\lambda(t_1)\prod_{i=1}^{n} s_\lambda(t_{i+1}-t_i\,)q(t-t_i)\mathrm{d}t_1\cdots\mathrm{d}t_n, s\le t,\end{cases}
\end{aligned}$$

and $s_\lambda(t)=\frac{\sin(l\sqrt{\lambda}t)}{l\sqrt{\lambda}}$. Note that for $0\le t\le1$,

$$\left|\frac{\sin(l\sqrt{\lambda}t)}{l\sqrt{\lambda}}\right|\le\frac{1}{l\left|\sqrt{\lambda}\right|}\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|t\right)\ \text{ and }\ \left|\frac{\sin(l\sqrt{\lambda}t)}{l\sqrt{\lambda}}\right|\le\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|t\right).$$

Thus when $s > t$,

$$\begin{aligned}
|S_n(s,t,\lambda)| &\le \frac{\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|(s-t)\right)}{l^2|\lambda|}\int_{0\le t_1\le\cdots\le t_{n+1}:=s-t}\prod_{i=1}^{n}|q(t_i+t)|\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\
&\le \frac{\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|(s-t)\right)}{l^2|\lambda|}\frac{\left(\int_0^{s-t}|q(u+t)|\,\mathrm{d}u\right)^n}{n!}\\
&\le \frac{\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|(s-t)\right)}{l^2|\lambda|}\frac{\|q\|_1^n}{n!}.
\end{aligned}$$

When $t \ge s$, through a similar process, one has

$$|S_n(s,t,\lambda)| \le \frac{\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|(t-s)\right)}{l^2|\lambda|}\frac{\|q\|_1^n}{n!}. \tag{3.9}$$

Therefore, (3.5) can be easily obtained from (3.8) and (3.9). Moreover, it follows from [18] that v_2 satisfies

$$\left|v_2(t,\lambda)-\frac{\sin(l\sqrt{\lambda}t)}{l\sqrt{\lambda}}\right| \le \frac{1}{l^2|\lambda|}\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|t+\|q\|_1\right). \tag{3.10}$$

This together with (3.4) and (3.5) yield that

$$\begin{aligned}
&|\Delta_1(\lambda)-\Delta_0(\lambda)|\\
\le\ & \frac{1}{l^2|\lambda|}\int_0^1\int_0^1 P(s)P(t)\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right||s-t|+\|q\|_1\right)\mathrm{d}\widetilde{\nu}_0(t)\mathrm{d}\widetilde{\nu}_1(s)\\
&+\frac{1}{l^2|\lambda|}P(1)\int_0^1 P(t)\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|(1-t)+\|q\|_1\right)\mathrm{d}\widetilde{\nu}_0(t)\\
&+\frac{1}{l^2|\lambda|}\int_0^1 P(t)\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|t+\|q\|_1\right)\mathrm{d}\widetilde{\nu}_1(t)\\
&+P(1)\frac{1}{l^2|\lambda|}\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|+\|q\|_1\right)\\
\le\ & C\frac{1}{l^2|\lambda|}\exp\left(l\left|\mathrm{Im}\sqrt{\lambda}\right|+\|q\|_1\right),
\end{aligned}$$

where $C = 4\exp\left(\int_0^1|p(t)|\,\mathrm{d}t\right)$. This completes the proof. □

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. For simplicity, denote $k=\sqrt{\lambda}$. In view of Lemma 2.6 and Remark 3.4, it is sufficient to prove that all the zeros of $\Delta_1(k^2)$ lie in a strip parallel to the real axis. Denote

$$\begin{aligned}
I_1 :&= \int_0^1 P(t)\sin(lkt)\,\mathrm{d}\widetilde{\nu}_1(t),\\
I_2 :&= P(1)\int_0^1 P(t)\sin(lk(1-t))\,\mathrm{d}\widetilde{\nu}_0(t),\\
I_3 :&= \int_0^1\int_0^1 P(s)P(t)\sin(lk(s-t))\,\mathrm{d}\widetilde{\nu}_0(t)\mathrm{d}\widetilde{\nu}_1(s).
\end{aligned}$$

Note that

$$lk\Delta_0(k^2) = I_3 - I_2 - I_1 + P(1)\sin(lk)\,.$$

To prove Theorem 3.1, we first show that

$$\begin{aligned} e^{-l|\mathrm{Im}k|}\left|lk\Delta_0(k^2)\right| &\geq e^{-l|\mathrm{Im}k|}\left(|P(1)|\left|\sin(lk)\right| - |I_1| - |I_2| - |I_3|\right) \\ &\geq \frac{|P(1)|}{2} - C\cdot\Omega\left(\mathrm{Im}k\right), \end{aligned} \tag{3.11}$$

where $C = 4\exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)$ and

$$\begin{aligned} \Omega\left(\mathrm{Im}k\right): &= e^{-2l|\mathrm{Im}k|} + \int_0^1 e^{l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_1(t) + \int_0^1 e^{-l|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_0(t) \\ &\quad + \int_0^1 e^{l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_0(t). \end{aligned}$$

In fact, since for each $z \in \mathbb{C}$,

$$\frac{e^{|\mathrm{Im}z|} - e^{-|\mathrm{Im}z|}}{2} \leq |\sin z| \leq \frac{e^{|\mathrm{Im}z|} + e^{-|\mathrm{Im}z|}}{2} \leq e^{|\mathrm{Im}z|},$$

we conclude that

$$\begin{aligned} |I_1| &= \left|\int_0^1 P(t)\sin(lkt)\,\mathrm{d}\widetilde{\nu}_1(t)\right| \leq \exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)\int_0^1 e^{l|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_1(t), \\ |I_2| &= \left|P(1)\int_0^1 P(t)\sin\left(lk\left(t-1\right)\right)\mathrm{d}\widetilde{\nu}_0(t)\right| \\ &\leq \exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)\int_0^1 e^{l|\mathrm{Im}k|(1-t)}\mathrm{d}\widetilde{\nu}_0(t) \\ |I_3| &= \left|\int_0^1\int_0^1 P(s)P(t)\sin\left(lk\left(s-t\right)\right)\mathrm{d}\widetilde{\nu}_0(t)\mathrm{d}\widetilde{\nu}_1(s)\right| \\ &\leq \frac{1}{2}\exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)\left|\int_0^1\int_0^1\left(e^{l|\mathrm{Im}k|(t-s)} + e^{-l|\mathrm{Im}k|(t-s)}\right)\mathrm{d}\widetilde{\nu}_0(t)\mathrm{d}\widetilde{\nu}_1(s)\right| \\ &\leq \frac{1}{2}\exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)\left[\int_0^1 e^{l|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_0(t) + \int_0^1 e^{l|\mathrm{Im}k|s}\mathrm{d}\widetilde{\nu}_1(s)\right], \end{aligned}$$

and

$$\begin{aligned} e^{-l|\mathrm{Im}k|}\left|P(1)\right|\left|\sin(lk)\right| &\geq e^{-l|\mathrm{Im}k|}\left|P(1)\right|\frac{e^{l|\mathrm{Im}k|} - e^{-l|\mathrm{Im}k|}}{2} \\ &\geq \frac{|P(1)|}{2} - \frac{|P(1)|\,e^{-2l|\mathrm{Im}k|}}{2}. \end{aligned}$$

Thus (3.11) can be obtained directly. Moreover, it can be shown that

$$\lim_{\mathrm{Im}k\to\infty}\Omega\left(\mathrm{Im}k\right) = 0. \tag{3.12}$$

In fact, note that $\widetilde{\nu}_i(t)$, $i = 0, 1$, are nondecreasing functions, and $\lim\limits_{t\to0^+}\widetilde{\nu}_i(t) = \widetilde{\nu}_i(0) = 0$, $\lim\limits_{t\to1^-}\widetilde{\nu}_i(t) = \widetilde{\nu}_i(1) = 1$. Thus given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\widetilde{\nu}_i(1) - \widetilde{\nu}_i(1-\delta) < \frac{\epsilon}{2}.$$

For such a $\delta > 0$, there exists a number $K_0 > 0$ such that if $|\mathrm{Im}k| > K_0$,

$$\begin{aligned}(3.13)\quad \int_0^1 e^{l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_i(t) &= \int_0^{1-\delta} e^{l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_i(t) + \int_{1-\delta}^1 e^{l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_i(t) \\ &\leq e^{-\delta l|\mathrm{Im}k|} + \widetilde{\nu}_i(1) - \widetilde{\nu}_i(1-\delta) \leq \epsilon,\ i = 0, 1.\end{aligned}$$

Thus

$$\lim_{\mathrm{Im}k\to\infty} \int_0^1 e^{l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_i(t) = 0,\ i = 0, 1.$$

Similarly, $\lim\limits_{\mathrm{Im}k\to\infty} \int_0^1 e^{-l|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_0(t) = 0$. This arrives the statement (3.12).

Thus from (3.11) and (3.12), it is obvious that there exists a number $K > 0$ such that if $|\mathrm{Im}k| > K$,

$$\begin{aligned}(3.14)\qquad e^{-l|\mathrm{Im}k|}\left|lk\Delta_0(k^2)\right| &\geq \frac{|P(1)|}{2} - C\cdot\Omega\left(\mathrm{Im}k\right) \\ &\geq \frac{|P(1)|}{4}.\end{aligned}$$

What should be noted is that the positive number K depends only on $|P(1)|$, $C = 4\exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)$, l, $\widetilde{\nu}_0$ and $\widetilde{\nu}_1$.

Now we define $E(k) := lk\Delta_1(k^2) - lk\Delta_0(k^2)$. Then it follows from Lemma 3.5 that

$$(3.15)\qquad |E(k)| \leq C\frac{1}{l\,|k|}\exp\left(l\,|\mathrm{Im}k| + \|q\|_1\right).$$

Suppose that k_0 is an arbitrary zero of $\Delta_1(k^2)$, i.e., k_0^2 is an eigenvalue of the problem (1.1) or (3.2). Then

$$(3.16)\qquad lk_0\Delta_1(k_0^2) = lk_0\Delta_0(k_0^2) + E(k_0) = 0.$$

Thus from (3.14), (3.15) and (3.16), if $|\mathrm{Im}k_0| > K$, we have

$$\begin{aligned}(3.17)\qquad |\mathrm{Im}k_0| \leq |k_0| &\leq C\exp\left(\|q\|_1\right)\frac{\exp\left(l\,|\mathrm{Im}k_0|\right)}{l^2\,|k_0\Delta_0(k_0^2)|} \\ &\leq C\exp\left(\|q\|_1\right)\frac{4}{l\,|P(1)|}.\end{aligned}$$

Hence $|\mathrm{Im}k_0| \leq \max\left\{C\exp\left(\|q\|_1\right)\frac{4}{l|P(1)|}, K\right\}$, which means that all the zeros of $\Delta_1(k^2)$ lie in a strip parallel to the real axis. This completes the proof. □

Remark 3.6. $\Delta_1(\lambda)$ is an entire function of order $\frac{1}{2}$. In fact, recall that for any $z \in \mathbb{C}$,

$$\frac{e^{|\mathrm{Im}z|} - e^{-|\mathrm{Im}z|}}{2} \leq |\sin z| \leq \frac{e^{|\mathrm{Im}z|} + e^{-|\mathrm{Im}z|}}{2} \leq e^{|\mathrm{Im}z|}.$$

Denote $M(r) := \max\left\{|\Delta_1(\lambda)| : |\lambda| = r\right\}$. Recall that $C = 4\exp\left(\int_0^1 |p(t)|\,\mathrm{d}t\right)$, thus it follows from Lemma 3.5 that

$$(3.18)\qquad M(r) \leq C\left(\frac{\exp\left(\|q\|_1\right)}{l^2 r} + \frac{1}{l\sqrt{r}}\right)e^{l\sqrt{r}}.$$

Moreover, for $\lambda=-r$, $r\in\mathbb{R}_+=(0,+\infty)$, from Lemma 3.5 and the proof of (3.11), one has

$$\begin{aligned}|\Delta_1(\lambda)| \quad\geq\quad & |P(1)|\frac{e^{l\sqrt{r}}-e^{-l\sqrt{r}}}{2l\sqrt{r}}-C\frac{\int_0^1 e^{l\sqrt{r}t}\mathrm{d}\widetilde{\nu}_0(t)}{l\sqrt{r}}-C\frac{\int_0^1 e^{l\sqrt{r}t}\mathrm{d}\widetilde{\nu}_1(t)}{l\sqrt{r}}\\ & -C\frac{\int_0^1 e^{l\sqrt{r}(1-t)}\mathrm{d}\widetilde{\nu}_0(t)}{l\sqrt{r}}-C\frac{\exp\left(\|q\|_1\right)}{l^2 r}e^{l\sqrt{r}}.\end{aligned}$$

This together with (3.18) yield

$$\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}=\frac{1}{2}.$$

Therefore, $\Delta_1(\lambda)$ is an entire function of order $\frac{1}{2}$ and thus has an infinite number of zeros ([19, Definition 2.1.1 and Theorem 2.9.2]).

4. Dependence of λ_m on the Coefficients a, b and Distributions ν_0, ν_1

In Section 2, we study the continuity of eigenvalues on the eigenvalue problem with respect to the topology induced by (1.4). In view of the meaning of λ_m introduced in Remark 3.2, in this section we turn to discuss the dependence of λ_m on the problem with respect to a stronger topology (see Theorem 1.1). Throughout the rest of this paper, ω represents a fixed problem (1.1), i.e.,

$$\omega:\begin{cases} a(x)y''(x)+b(x)y'(x)=\lambda y(x),\ x\in(0,1),\\ y(0)=\int_0^1 y(x)\mathrm{d}\nu_0(x),\ y(1)=\int_0^1 y(x)\mathrm{d}\nu_1(x).\end{cases}$$

Consider the problems ω_n, $n\in\mathbb{N}$, as follows,

$$\omega_n:\begin{cases} a_n(x)y''(x)+b_n(x)y'(x)=\lambda y(x),\ x\in(0,1),\\ y(0)=\int_0^1 y(x)\mathrm{d}\nu_{0,n}(x),\ y(1)=\int_0^1 y(x)\mathrm{d}\nu_{1,n}(x),\end{cases}$$

where $\nu_{i,n}$, $i=0,1$, are probability distributions on the interval $(0,1)$, and the coefficients a_n, b_n all satisfy the condition (1.3).

Similar to the notations related to the problem ω in Section 3, now we introduce some notations for the problem ω_n. Denote

$$p_n(t)=\frac{l_n\left(b_n(x)-\frac{1}{2}a_n'(x)\right)}{\sqrt{-a_n(x)}},\ \widetilde{\nu}_{0,n}(t)=\nu_{0,n}(x),\ \widetilde{\nu}_{1,n}(t)=\nu_{1,n}(x) \tag{4.1}$$

where $t=l_n^{-1}\cdot\int_0^x\frac{\mathrm{d}s}{\sqrt{-a_n(s)}}$ and $l_n:=\int_0^1\frac{\mathrm{d}s}{\sqrt{-a_n(s)}}$. Furthermore, denote

$$\begin{aligned} q_n(t) &= \frac{1}{4}p_n^2(t)-\frac{1}{2}p_n'(t),\ C_n=4\exp\left(\int_0^1|p_n(t)|\,\mathrm{d}t\right),\\ P_n(t) &= \exp\left(\frac{\int_0^t p_n(s)\mathrm{d}s}{2}\right).\end{aligned} \tag{4.2}$$

Corresponding to $\Delta_1(\lambda)$ and $\Delta_0(\lambda)$ related to the problem ω, $\Delta_{1,n}(\lambda)$ and $\Delta_{0,n}(\lambda)$ represent the analogous objects related to the problem ω_n, respectively.

Let us mention that the proof of Theorem 1.1, which will be given at the end of this section, relies heavily on the following result (Lemma 4.1). In Section 3, by analyzing the equivalent eigenvalue problem (3.2), we show that all the eigenvalues of the problem ω are confined to a parabolic neighborhood of the real axis.

The following statement illustrates the change of the neighborhood under "small" perturbations.

Lemma 4.1. *If* $\|a_n - a\|_{W^{2,2}} \to 0, \|b_n - b\|_{W^{1,2}} \to 0,\ \nu_{0,n} \xrightarrow{w} \nu_0,\ \nu_{1,n} \xrightarrow{w} \nu_1,$ *as* $n \to \infty$, *then there exists a constant* R *such that all the eigenvalues of the problems* ω *and* ω_n, $n \in \mathbb{N}$ *lie in the region*

$$\Lambda = \left\{ \lambda \in \mathbb{C} \,\middle|\, Re\lambda > \frac{(Im\lambda)^2}{4R^2} - R^2 \right\}.$$

Remark 4.2. We say $\nu_{i,n} \xrightarrow{w} \nu_i$, i.e., $\nu_{i,n}$ is weakly convergent to $\nu_i, i = 0, 1$, iff, for each $f \in C(J, \mathbb{R})$, one has

$$\lim_{n\to\infty} \int_0^1 f(x) \mathrm{d}\nu_{i,n}(x) = \int_0^1 f(x) \mathrm{d}\nu_i(x).$$

Note that from [20, Example 2.1], it follows that for $i = 0, 1$, $\nu_{i,n} \xrightarrow{w} \nu_i$ if and only if $\lim\limits_{n\to\infty} \nu_{i,n}(x) = \nu_i(x)$ for each continuous point x of $\nu_i(x)$.

Remark 4.3. The above remark and Dominated Convergence Theorem ([21, Theorem 2.8.1]) imply that if $\nu_{i,n} \xrightarrow{w} \nu_i$, one has

$$\lim_{n\to\infty} \int_0^1 |\nu_{i,n}(x) - \nu_i(x)| \, \mathrm{d}x = 0.$$

To prove Lemma 4.1, the following result is also needed.

Lemma 4.4. *If* $\|a_n - a\|_{W^{2,2}} \to 0, \|b_n - b\|_{W^{1,2}} \to 0,$ *as* $n \to \infty$, *then*

$$\|p_n - p\|_{W^{1,2}} \to 0, \quad \|q_n - q\|_1 \to 0, \quad \text{as } n \to \infty.$$

Proof. **Step 1**: We first give some basic facts which will be used in Step 2. Firstly, since $\|a_n - a\|_{W^{2,2}} \to 0$ and $\|b_n - b\|_{W^{1,2}} \to 0$, it follows from the Sobolev embedding theorem that

$$\max_{x\in[0,1]} |a_n(x) - a(x)| \to 0, \quad \max_{x\in[0,1]} |a_n'(x) - a'(x)| \to 0, \text{ as } n \to \infty, \tag{4.3}$$

and

$$\max_{x\in[0,1]} |b_n(x) - b(x)| \to 0, \text{ as } n \to \infty. \tag{4.4}$$

Next, for each $x \in [0,1]$, $n \in \mathbb{N}$, there exists a unique number $f_n(x) \in [0,1]$ such that

$$l^{-1} \cdot \int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}} = l_n^{-1} \cdot \int_0^{f_n(x)} \frac{\mathrm{d}s}{\sqrt{-a_n(s)}} \tag{4.5}$$

and $f_n(0) = 0$, $f_n(1) = 1$. Implicit functions theorem also shows that for each $n \in \mathbb{N}$, $f_n(x)$ is differentiable in $x \in [0,1]$ and $f_n'(x) = \frac{l_n\sqrt{-a_n(f_n(x))}}{l\sqrt{-a(x)}}$. Furthermore, based on the equality (4.5) , one has

$$
\begin{aligned}
&\min_{x\in[0,1]}\left\{\frac{1}{\sqrt{-a(x)}}\right\}\cdot|f_n(x)-x|\\
\leq\ & \left|\int_0^{f_n(x)}\frac{\mathrm{d}s}{\sqrt{-a(s)}}-\int_0^{x}\frac{\mathrm{d}s}{\sqrt{-a(s)}}\right|\\
\leq\ & \left|\int_0^{f_n(x)}\frac{\mathrm{d}s}{\sqrt{-a(s)}}-\int_0^{f_n(x)}\frac{\mathrm{d}s}{\sqrt{-a_n(s)}}\right|+\left|\frac{l_n}{l}\int_0^{x}\frac{\mathrm{d}s}{\sqrt{-a(s)}}-\int_0^{x}\frac{\mathrm{d}s}{\sqrt{-a(s)}}\right|\\
\leq\ & 2\int_0^1\left|\frac{1}{\sqrt{-a(s)}}-\frac{1}{\sqrt{-a_n(s)}}\right|\mathrm{d}s.
\end{aligned}
$$

Hence it follows from (4.3) that

$$
\max_{x\in[0,1]}|f_n(x)-x|\to 0,\ \text{ as } n\to\infty. \tag{4.6}
$$

In view of (4.3) and (4.6) , it is easy to see that

$$
\max_{x\in[0,1]}|a_n(f_n(x))-a(x)|\to 0,\ l_n\to l,\ \text{ as } n\to\infty.
$$

Therefore, there exists a positive constant M_1 such that for all $n\in\mathbb{N}$ and $x\in[0,1]$,

$$
\frac{1}{f_n'(x)}=\frac{l\sqrt{-a(x)}}{l_n\sqrt{-a_n(f_n(x))}}<M_1. \tag{4.7}
$$

Step 2: Based on the above facts, now we turn to prove

$$
\|p_n-p\|_{W^{1,2}}\to 0,\ \text{ as } n\to\infty. \tag{4.8}
$$

Indeed, denote $M_0:=\max\limits_{x\in[0,1]}\left\{\frac{1}{l\sqrt{-a(x)}}\right\}$. Then

$$
\begin{aligned}
&\int_0^1|p_n(t)-p(t)|^2\,\mathrm{d}t \qquad (4.9)\\
=\ & \int_0^1\frac{1}{l\sqrt{-a(x)}}\left|p_n\left(l_n^{-1}\cdot\int_0^{f_n(x)}\frac{\mathrm{d}s}{\sqrt{-a_n(s)}}\right)-p\left(l^{-1}\cdot\int_0^{x}\frac{\mathrm{d}s}{\sqrt{-a(s)}}\right)\right|^2\mathrm{d}x\\
\leq\ & M_0\int_0^1\left|\frac{l_n\left[b_n\left(f_n(x)\right)-\frac12 a_n'\left(f_n(x)\right)\right]}{\sqrt{-a_n(f_n(x))}}-\frac{l\left(b(x)-\frac12 a'(x)\right)}{\sqrt{-a(x)}}\right|^2\mathrm{d}x\\
\leq\ & 2M_0\int_0^1\left|\frac{l_nb_n\left(f_n(x)\right)}{\sqrt{-a_n(f_n(x))}}-\frac{lb(x)}{\sqrt{-a(x)}}\right|^2\mathrm{d}x\\
&+\frac{M_0}{2}\int_0^1\left|\frac{l_na_n'\left(f_n(x)\right)}{\sqrt{-a_n(f_n(x))}}-\frac{la'(x)}{\sqrt{-a(x)}}\right|^2\mathrm{d}x.
\end{aligned}
$$

Moreover, a straightforward calculation yields

$$
p_n'(t)=l_n^2\left(b_n'(x)-\frac12 a_n''(x)\right)-\frac{l_n^2}{2}\frac{\left(b_n(x)-\frac12 a_n'(x)\right)a_n'(x)}{a_n(x)}
$$

where $t = l_n^{-1} \cdot \int_0^x \frac{\mathrm{d}s}{\sqrt{-a_n(s)}}$. Thus

$$
\begin{aligned}
(4.10)\quad & \int_0^1 \left|p_n'(t) - p'(t)\right|^2 \mathrm{d}t \\
\leq\ & 4M_0 \int_0^1 \left|l_n^2 b_n'(f_n(x)) - l^2 b'(x)\right|^2 \mathrm{d}x + M_0 \int_0^1 \left|l_n^2 a_n''(f_n(x)) - l^2 a''(x)\right|^2 \mathrm{d}x \\
& + M_0 \int_0^1 \left|\frac{l_n^2 b_n(f_n(x)) a_n'(f_n(x))}{a_n(f_n(x))} - \frac{l^2 b(x) a'(x)}{a(x)}\right|^2 \mathrm{d}x \\
& + \frac{M_0}{4} \int_0^1 \left|\frac{l_n^2 \left(a_n'(f_n(x))\right)^2}{a_n(f_n(x))} - \frac{l^2 \left(a'(x)\right)^2}{a(x)}\right|^2 \mathrm{d}x.
\end{aligned}
$$

In order to prove (4.8), we only aim to show

$$
\lim_{n\to\infty} \int_0^1 \left|l_n^2 b_n'(f_n(x)) - l^2 b'(x)\right|^2 \mathrm{d}x = 0, \tag{4.11}
$$

since other terms can be dealt with in a similar way. In fact, in view of (4.7), we have

$$
\begin{aligned}
& \int_0^1 \left|l_n^2 b_n'(f_n(x)) - l^2 b'(x)\right|^2 \mathrm{d}x \\
\leq\ & 3M_1 \int_0^1 \left|l_n^2 b_n'(f_n(x)) - l_n^2 b'(f_n(x))\right|^2 f_n'(x)\mathrm{d}x + 3 \int_0^1 \left|l^2 b'(f_n(x)) - l^2 b'(x)\right|^2 \mathrm{d}x \\
& + 3M_1 \int_0^1 \left|l_n^2 b'(f_n(x)) - l^2 b'(f_n(x))\right|^2 f_n'(x)\mathrm{d}x \\
\leq\ & 3M_1 l_n^4 \int_0^1 \left|b_n'(u) - b'(u)\right|^2 \mathrm{d}u + 3l^2 \int_0^1 \left|b'(f_n(x)) - b'(x)\right|^2 \mathrm{d}x \\
& + 3\left|l_n^2 - l^2\right|^2 M_1 \int_0^1 \left|b'(u)\right|^2 \mathrm{d}u.
\end{aligned}
$$

It is obvious that

$$
\lim_{n\to\infty} \int_0^1 \left|b_n'(u) - b'(u)\right|^2 \mathrm{d}u = 0 \text{ and } \lim_{n\to\infty} \left|l_n^2 - l^2\right| = 0.
$$

Therefore, in order to prove (4.11), we just prove

$$
\lim_{n\to\infty} \int_0^1 \left|b'(f_n(x)) - b'(x)\right|^2 \mathrm{d}x = 0.
$$

Note that $b' \in L^2(J, \mathbb{R})$, thus given any $\epsilon > 0$, there exists a continuous function φ on $[0, 1]$ such that

$$
\int_0^1 \left|\varphi(x) - b'(x)\right|^2 \mathrm{d}x < \epsilon. \tag{4.12}
$$

This together with (4.6) yield that for the above arbitrary $\epsilon > 0$, there exists a number $N_1 > 0$ such that if $n > N_1$,

$$\begin{aligned}
& \int_0^1 \left|b'(f_n(x)) - b'(x)\right|^2 \mathrm{d}x \\
\leq\ & 3M_1 \int_0^1 \left|b'(f_n(x)) - \varphi(f_n(x))\right|^2 f_n'(x)\mathrm{d}x + 3\int_0^1 \left|\varphi(f_n(x)) - \varphi(x)\right|^2 \mathrm{d}x \\
& +3\int_0^1 \left|\varphi(x) - b'(x)\right|^2 \mathrm{d}x \\
\leq\ & 3M_1 \int_0^1 \left|b'(u) - \varphi(u)\right|^2 \mathrm{d}u + 3\int_0^1 \left|\varphi(f_n(x)) - \varphi(x)\right|^2 \mathrm{d}x + 3\int_0^1 \left|\varphi(x) - b'(x)\right|^2 \mathrm{d}x \\
<\ & 3\,(M_1+2)\,\epsilon.
\end{aligned}$$

This proves (4.11), thus (4.8) can be obtained. Now the statement

$$\|q_n - q\|_1 \to 0, \text{ as } n \to \infty$$

directly follows from (4.8) and

$$\int_0^1 |q_n(t) - q(t)|\,\mathrm{d}t \leq \frac{1}{4}\int_0^1 \left|p_n^2(t) - p^2(t)\right|\mathrm{d}t + \frac{1}{2}\int_0^1 |p_n'(t) - p'(t)|\,\mathrm{d}t.$$

Therefore, the assertion of Lemma 4.4 is proved. □

Now we are in a position to prove Lemma 4.1.

Proof of Lemma 4.1. Firstly, it follows from Lemma 4.4 that there exists a number $N_0 > 0$ such that if $n > N_0$,

$$C_n \leq C+1,\ l_n \geq \frac{l}{2},\ \|q_n\|_1 \leq \|q\|_1 + 1,\ |P_n(1)| \geq \frac{|P(1)|}{2}. \tag{4.13}$$

Note that all the notations here can be found in (4.1), (4.2) and Section 3. Next, we will show that there exists a number $N_0' > 0$, which is independent of $k \in \mathbb{C}$, such that if $n > N_0'$,

$$\int_0^1 e^{-l_n|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_{0,n}(t) \leq \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_0(t) + \frac{|P(1)|}{24\,(C+1)}, \tag{4.14}$$

$$\int_0^1 e^{l_n|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_{i,n}(t) \leq \int_0^1 e^{\frac{1}{2}l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_i(t) + \frac{|P(1)|}{24\,(C+1)}, i=0,1. \tag{4.15}$$

In fact, in view of $\max\limits_{x\in[0,1]} |a_n(x) - a(x)| \to 0$, as $n \to \infty$, there exists a number $N_1 > 0$, which is independent of x, such that if $n > N_1$,

$$\begin{aligned}
& \left|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a_n(s)}} - \int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}\right| \\
=\ & \left|\int_0^x \left(\frac{\sqrt{-a(s)}}{\sqrt{-a_n(s)}} - 1\right)\frac{1}{\sqrt{-a(s)}}\mathrm{d}s\right| \leq \frac{1}{2}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}
\end{aligned}$$

and similarly,

$$\left|\int_x^1 \frac{\mathrm{d}s}{\sqrt{-a_n(s)}} - \int_x^1 \frac{\mathrm{d}s}{\sqrt{-a(s)}}\right| \leq \frac{1}{2}\int_x^1 \frac{\mathrm{d}s}{\sqrt{-a(s)}}.$$

Therefore, if $n > N_1$,

$$\begin{aligned}
\int_0^1 e^{-l_n|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_{0,n}(t) &= \int_0^1 e^{-l_n|\mathrm{Im}k|l_n^{-1}\cdot\int_0^x \frac{\mathrm{d}s}{\sqrt{-a_n(s)}}}\mathrm{d}\nu_{0,n}(x)\\
&\leq \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_{0,n}(x),\\
\int_0^1 e^{l_n|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_{i,n}(t) &= \int_0^1 e^{-l_n|\mathrm{Im}k|l_n^{-1}\cdot\int_x^1 \frac{\mathrm{d}s}{\sqrt{-a_n(s)}}}\mathrm{d}\nu_{i,n}(x)\\
&\leq \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|l^{-1}\int_x^1 \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_{i,n}(x)\\
&= \int_0^1 e^{\frac{l}{2}|\mathrm{Im}k|\left(l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}-1\right)}\mathrm{d}\nu_{i,n}(x),\ i=0,1.
\end{aligned}$$

Thus in order to prove (4.14) and (4.15), it is sufficient to show there exists a number $N_2 > 0$, which is independent of $k \in \mathbb{C}$, such that if $n > N_2$,

$$\begin{aligned}
&\int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_{0,n}(x) \\
&\leq \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_0(x) + \frac{|P(1)|}{24\,(C+1)},
\end{aligned}\tag{4.16}$$

$$\begin{aligned}
&\int_0^1 e^{\frac{l}{2}|\mathrm{Im}k|\left(l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}-1\right)}\mathrm{d}\nu_{i,n}(x)\\
&\leq \int_0^1 e^{\frac{l}{2}|\mathrm{Im}k|\left(l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}-1\right)}\mathrm{d}\nu_i(x) + \frac{|P(1)|}{24\,(C+1)},\ i=0,1.
\end{aligned}\tag{4.17}$$

Since $\lim\limits_{x\to 0^+}\nu_0(x) = 0$, there exists a continuous point $\delta \in (0,1)$ of $\nu_0(x)$ such that

$$\nu_0(\delta) < \frac{|P(1)|}{100\,(C+1)}.\tag{4.18}$$

Moreover, since $\nu_{0,n} \overset{w}{\to} \nu_0$, Remark 4.2 implies that there exists a number $N_3 > 0$ such that if $n > N_3$ one has

$$|\nu_{0,n}(\delta) - \nu_0(\delta)| < \frac{|P(1)|}{100\,(C+1)},\tag{4.19}$$

thus

$$\nu_{0,n}(\delta) < \frac{|P(1)|}{50\,(C+1)}.\tag{4.20}$$

Denote $M := \max\limits_{x\in[0,1]}\left\{\frac{1}{\sqrt{-a(x)}}\right\}$. Note that $\lim\limits_{|\mathrm{Im}k|\to\infty}\frac{|\mathrm{Im}k|M}{2}e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^\delta \frac{\mathrm{d}s}{\sqrt{-a(s)}}} = 0$ for the fixed number δ. Thus there exists a number M_δ such that for all $k \in \mathbb{C}$,

$$\frac{1}{2}e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^\delta \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\,|\mathrm{Im}k|\,M < M_\delta.\tag{4.21}$$

Hence from (4.18)–(4.21), Remark 4.3 and the integration by parts formula, it follows that there exists a number $N_4 > N_3$, which is independent of $k \in \mathbb{C}$, such that if $n > N_4$,

$$\begin{aligned}
& \left|\int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_{0,n}(x) - \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|l^{-1}\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_0(x)\right| \\
\leq & \left|\int_0^\delta e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_{0,n}(x)\right| + \left|\int_0^\delta e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_0(x)\right| \\
& + \left|\int_\delta^1 e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_{0,n}(x) - \int_\delta^1 e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}\nu_0(x)\right| \\
\leq & \ \nu_{0,n}(\delta) + \nu_0(\delta) + |\nu_{0,n}(\delta) - \nu_0(\delta)| \\
& + \frac{1}{2}|\mathrm{Im}k|\, M \left|\int_\delta^1 \nu_{0,n}(x) e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}} - \nu_0(x) e^{-\frac{1}{2}|\mathrm{Im}k|\int_0^x \frac{\mathrm{d}s}{\sqrt{-a(s)}}}\mathrm{d}x\right| \\
\leq & \ \nu_{0,n}(\delta) + \nu_0(\delta) + |\nu_{0,n}(\delta) - \nu_0(\delta)| + M_\delta \int_0^1 |\nu_{0,n}(x) - \nu_0(x)|\,\mathrm{d}x \\
< & \ \frac{|P(1)|}{24\,(C+1)}.
\end{aligned}$$

This arrives (4.16). (4.17) can be proved in a similar way. Now we complete the proof of (4.14) and (4.15).

Denote $N := \max\{N_0, N_0'\}$. Then from (3.11), (4.13), (4.14) and (4.15), we conclude that if $n > N$,

$$\begin{aligned}
e^{-l_n|\mathrm{Im}k|}\left|l_n k \Delta_{0,n}(k^2)\right| & \geq \frac{|P_n(1)|}{2} - C_n \cdot \Omega_n(\mathrm{Im}k) \\
& \geq \frac{|P(1)|}{4} - (C+1)\,\widetilde{\Omega}(\mathrm{Im}k) \\
& \geq \frac{|P(1)|}{8} - (C+1)\,\widehat{\Omega}(\mathrm{Im}k)
\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
\Omega_n(\mathrm{Im}k) : &= e^{-2l_n|\mathrm{Im}k|} + \int_0^1 e^{l_n|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_{1,n}(t) + \int_0^1 e^{-l_n|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_{0,n}(t) \\
& \quad + \int_0^1 e^{l_n|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_{0,n}(t), \\
\widetilde{\Omega}(\mathrm{Im}k) : &= e^{-l|\mathrm{Im}k|} + \int_0^1 e^{\frac{1}{2}l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_1(t) + \frac{|P(1)|}{24\,(C+1)} \\
& \quad + \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_0(t) + \frac{|P(1)|}{24\,(C+1)} \\
& \quad + \int_0^1 e^{\frac{1}{2}l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_0(t) + \frac{|P(1)|}{24\,(C+1)}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\Omega}(\mathrm{Im}k) : &= e^{-l|\mathrm{Im}k|} + \int_0^1 e^{\frac{1}{2}l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_1(t) + \int_0^1 e^{-\frac{l}{2}|\mathrm{Im}k|t}\mathrm{d}\widetilde{\nu}_0(t) \\
& \quad + \int_0^1 e^{\frac{1}{2}l|\mathrm{Im}k|(t-1)}\mathrm{d}\widetilde{\nu}_0(t).
\end{aligned}$$

Note that the positive number N is independent of k. In view of (3.12), we can also obtain

$$\lim_{\mathrm{Im}k\to\infty}\widehat{\Omega}\left(\mathrm{Im}k\right)=0.$$

This together with (4.22) yield that there exists a number $K_1>0$, which is independent of n, such that if $|\mathrm{Im}k|>K_1$,

$$\begin{aligned} e^{-l_n|\mathrm{Im}k|}\left|l_nk\Delta_{0,n}(k^2)\right| &\geq \frac{|P(1)|}{8}-(C+1)\,\widehat{\Omega}\left(\mathrm{Im}k\right) \\ &\geq \frac{|P(1)|}{16},\ n>N. \end{aligned} \tag{4.23}$$

Note that K_1 depends only $|P(1)|$, $C=4\exp\left(\int_0^1|p(t)|\,\mathrm{d}t\right)$, l, $\widetilde{\nu}_0$ and $\widetilde{\nu}_1$.

Now suppose that $k_{0,n}$ is an arbitrary zero of $\Delta_{1,n}(k^2)$, $n>N$, i.e., $k_{0,n}^2$ is an eigenvalue of the problem ω_n, $n>N$. Then from (3.17), (4.13) and (4.23), we conclude that if $|\mathrm{Im}k_{0,n}|>K_1$,

$$\begin{aligned} |\mathrm{Im}k_{0,n}|\leq|k_{0,n}| &\leq C_n\exp\left(\|q_n\|_1\right)\frac{\exp\left(l_n\left|\mathrm{Im}k_{0,n}\right|\right)}{l_n^2\left|k_{0,n}\Delta_{0,n}(k_{0,n}^2)\right|} \\ &\leq (C+1)\exp\left(\|q\|_1+1\right)\frac{32}{l\left|P(1)\right|}. \end{aligned}$$

Thus if $n>N$, $|\mathrm{Im}k_{0,n}|\leq\max\left\{(C+1)\exp\left(\|q\|_1+1\right)\frac{32}{l|P(1)|},K_1\right\}$. Note that the maximum depends only on the fixed problem ω. This together with Theorem 3.1 yield the statement of Lemma 4.1. □

As a consequence of Lemma 4.1, Theorem 1.1 announced in the introduction can be established.

Proof of Theorem 1.1. (1) Let $j\geq1$ be an arbitrary integer and fix a number $r\in(\mathrm{Re}\lambda_{k_j-1}^{j-1}(\omega),\mathrm{Re}\lambda_{k_j}^{j}(\omega))$. Denote

$$\Pi:=\left\{\lambda\in\mathbb{C}\,\middle|\,\mathrm{Re}\lambda>\frac{\left(\mathrm{Im}\lambda\right)^2}{4R^2}-R^2\text{ and }\mathrm{Re}\lambda<r\right\},$$

where R is the constant obtained in Lemma 4.1. Then Lemma 4.1 implies that the problem ω has exactly k_j eigenvalues, counted with algebraic multiplicities, in the open region Π and none on the boundary of Π. Thus by Remark 2.14, when n is sufficiently large, each problem ω_n has exactly k_j eigenvalues, counted with algebraic multiplicities, in the region Π. Moreover, it follows from Lemma 4.1 that these k_j eigenvalues are the first k_j.

Now fix numbers $r_l\in(\mathrm{Re}\lambda_{k_l-1}^{l-1}(\omega),\ \mathrm{Re}\lambda_{k_l}^{l}(\omega))$, $l=1,2,\ldots,j-1$, and separate eigenvalues of the problem ω with different real parts by small open regions Π_l in Π, where

$$\begin{aligned} \Pi_1 &:= \left\{\lambda\in\Pi\,|\,\mathrm{Re}\lambda<r_1\right\}, \\ \Pi_l &:= \left\{\lambda\in\Pi\,|\,\mathrm{Re}\lambda>r_{l-1}\text{ and }\mathrm{Re}\lambda<r_l\right\},\ l=2,3,\ldots,j-1, \\ \Pi_j &:= \left\{\lambda\in\Pi\,|\,\mathrm{Re}\lambda>r_{j-1}\text{ and }\mathrm{Re}\lambda<r\right\}. \end{aligned}$$

Then by applying Remark 2.14 to these open regions Π_l, we see that when n is sufficiently large, each problem ω_n has k_l-k_{l-1} eigenvalues, counted with algebraic multiplicities, in each region Π_l, $l = 1, 2, \ldots, j$. Therefore, the statement (1) of Theorem 1.1 can be directly obtained from Theorem 2.11.

(2) The statement (2) of Theorem 1.1 is a direct consequence of the statement (1). In fact, in view of statement (1), for each $j \in \mathbb{N}_0$, given any $\epsilon > 0$, there exists a number $N > 0$ such that if $n > N$, one has

$$\sum_{m=k_j}^{k_{j+1}-1} \left|\mathrm{Re}\lambda_{m_n}(\omega_n) - \mathrm{Re}\lambda_m^j(\omega)\right| < \sum_{m=k_j}^{k_{j+1}-1} \left|\lambda_{m_n}(\omega_n) - \lambda_m^j(\omega)\right| < \epsilon, \tag{4.24}$$

where the set $\{m_n : m = k_j, \cdots, k_{j+1} - 1\} = \{m : m = n_j, \cdots, n_{j+1} - 1\}$. Since

$$\mathrm{Re}\lambda^j_{k_j}(\omega) = \mathrm{Re}\lambda^j_{k_j+1}(\omega) = \cdots = \mathrm{Re}\lambda^j_{k_{j+1}-1}(\omega),$$

it follows from (4.24) that if $n > N$, one has

$$\sum_{m=k_j}^{k_{j+1}-1} \left|\mathrm{Re}\lambda_m(\omega_n) - \mathrm{Re}\lambda_m^j(\omega)\right| < \epsilon.$$

Note that $\lambda_m^j(\omega)$ is the m-th eigenvalue of ω. This arrives the statement (2) of Theorem 1.1. $\square$

Theorem 1.1 shows that $\mathrm{Re}\lambda_m$ depends continuously on the coefficients a, b and distributions ν_0, ν_1 with respect to a stronger topology than that induced by (1.4). As consequences of Theorem 1.1, one can deduce the following results (see Corollary 4.5 and Corollary 4.7), which illustrates that the continuity of λ_m on the coefficients a, b and distributions ν_0, ν_1 could be guaranteed in some special situations.

Corollary 4.5. Suppose that $\|a_n - a\|_{W^{2,2}} \to 0, \|b_n - b\|_{W^{1,2}} \to 0$, $\nu_{0,n} \overset{w}{\to} \nu_0$, $\nu_{1,n} \overset{w}{\to} \nu_1$, as $n \to \infty$.

(1) Let $\lambda_{m_0}(\omega)$, $m_0 \in \mathbb{N}$, be a real eigenvalue of ω with algebraic multiplicity $k + 1$. Suppose that

$$\lambda_{m_0}(\omega) = \lambda_{m_0+1}(\omega) = \cdots = \lambda_{m_0+k}(\omega),$$
$$\mathrm{Re}\lambda_{m_0-1}(\omega) < \mathrm{Re}\lambda_{m_0}(\omega) < \mathrm{Re}\lambda_{m_0+k+1}(\omega),$$

then as $n \to \infty$, one has

$$\lambda_m(\omega_n) \to \lambda_m(\omega),\ m = m_0, m_0 + 1, \ldots, m_0 + k.$$

(2) Let $\lambda_{m_0}(\omega)$, $m_0 \in \mathbb{N}$, be an algebraically simple eigenvalue of ω with negative imaginary part. If

$$\mathrm{Re}\lambda_{m_0-1}(\omega) < \mathrm{Re}\lambda_{m_0}(\omega) = \mathrm{Re}\lambda_{m_0+1}(\omega) < \mathrm{Re}\lambda_{m_0+2}(\omega),$$

then for $m = m_0, m_0 + 1$, one has $\lambda_m(\omega_n) \to \lambda_m(\omega)$ as $n \to \infty$.

Remark 4.6. Note that Remark 2.8 implies that in the above statement, $\lambda_{m_0+1}(\omega) = \overline{\lambda_{m_0}(\omega)}$ and $\lambda_{m_0+1}(\omega_n) = \overline{\lambda_{m_0}(\omega_n)}$ for sufficiently large n.

Corollary 4.7. Let $\lambda_m(\nu_0, \nu_1)$ denote the m-th eigenvalue of the problem (1.1) with constant coefficients $a \equiv -1$, $b \equiv 0$. Then for each $m \in \mathbb{N}_0$, $\lambda_m(\nu_0, \nu_1)$ is continuous in the probability distributions ν_0, ν_1 with respect to the weak topology.

Proof. In the case of $a \equiv -1$, $b \equiv 0$, it follows from [6] that all the eigenvalues of the problem (1.1) are real. Thus the statement of the corollary can be directly obtained from Theorem 1.1. $\square$

To conclude the discussion of this section, we give an example to show that the continuous eigenvalue branch in the sense of Theorem 2.11 is not necessarily determined by a fixed index.

Example 4.8. Consider the eigenvalue problem with constant coefficients $a < 0$, $b \in \mathbb{R}$ and $\nu_0 = \nu_1 = \delta_{\frac{1}{2}}$, i.e.,

$$\begin{cases} ay''(x) + by'(x) = \lambda y(x), \ x \in (0,1), \\ y(0) = y(\frac{1}{2}) = y(1). \end{cases} \tag{4.25}$$

Under the transformation $v(x) = \exp\left(\frac{bx}{2a}\right) y(x)$, problem (4.25) is equivalent to the following eigenvalue problem

$$\begin{cases} -v''(x) + qv(x) = -\frac{\lambda}{a} v(x), \ x \in (0,1), \\ v(0) = Av(\frac{1}{2}) = \ A^2 v(1), \end{cases} \tag{4.26}$$

where $q = \frac{1}{4}\left(\frac{b}{a}\right)^2$ and $A = \exp\left(\frac{b}{-4a}\right)$. By Lemma 3.3, it is easy to see that λ is an eigenvalue of the problem (4.25) or (4.26) if and only if

$$\Delta_1(\lambda) = \det \begin{pmatrix} A\cos\left(\frac{\sqrt{-\frac{\lambda}{a}-q}}{2}\right) - 1 & A\frac{\sin\left(\frac{\sqrt{-\frac{\lambda}{a}-q}}{2}\right)}{\sqrt{-\frac{\lambda}{a}-q}} \\ 1 - A^2\cos\sqrt{-\frac{\lambda}{a} - q} & -A^2\frac{\sin\sqrt{-\frac{\lambda}{a}-q}}{\sqrt{-\frac{\lambda}{a}-q}} \end{pmatrix} = 0. \tag{4.27}$$

Denote the eigenvalues of the problem (4.25) by $\lambda_m(a,b)$, $m \in \mathbb{N}_0$. Assume $b > 0$ and $|b| < -4\sqrt{3}a\pi$, then each eigenvalue is algebraically simple, and all the eigenvalues can be ordered as follows according to Remark 3.2,

$$\begin{aligned} &\lambda_0(a,b) = 0, \ \lambda_{4n+1}(a,b) = -4a\,(2n+1)^2\pi^2 - \frac{b^2}{4a}, \\ &\lambda_{4n+2}(a,b) = -16a\,(n+1)^2\pi^2 - 2b\,(n+1)\,\pi i, \\ &\lambda_{4n+3}(a,b) = -16a\,(n+1)^2\pi^2 + 2b\,(n+1)\,\pi i, \\ &\lambda_{4n+4}(a,b) = -16a\,(n+1)^2\pi^2 - \frac{b^2}{4a}, \ n \in \mathbb{N}_0. \end{aligned}$$

Note that when $b < 0$ and $|b| < -4\sqrt{3}a\pi$, to obtain the arrangement of eigenvalues, we only need to exchange the order of λ_{4n+2} and λ_{4n+3} in the above result. Moreover, when $b = 0$, direct calculation yields that

$$\begin{aligned} &\lambda_0(-1,0) = 0, \\ &\lambda_{4n+1}(-1,0) = -4a\,(2n+1)^2\pi^2, \ \lambda_{4n+2}(-1,0) = -16a\,(n+1)^2\pi^2, \\ &\lambda_{4n+3}(-1,0) = -16a\,(n+1)^2\pi^2, \ \lambda_{4n+4}(-1,0) = -16a\,(n+1)^2\pi^2, \ n \in \mathbb{N}_0. \end{aligned}$$

From the above results, it is obvious that for each $m \in \mathbb{N}_0$, as a function of $(a,b) \in (-\infty,0) \times \mathbb{R}$, $\lambda_m(a,b)$ is continuous on the region

$$\Gamma_1 := \left\{ (a,b) \in (-\infty,0) \times \mathbb{R} |\, |b| < -4\sqrt{3}a\pi \right\}.$$

This is consistent with the statement of Corollary 4.5.

However, denote $\Gamma_2 := \left\{ (a,b) \in (-\infty,0) \times \mathbb{R} |\, |b| = -4\sqrt{3}a\pi \right\}$, then it is easy to see that λ_1 and λ_2 are discontinuous at each point of Γ_2. In fact, when $b = -4\sqrt{3}a\pi$, all the eigenvalues can be ordered as follows,

$$\lambda_0(a,b)=0,\ \lambda_1(a,b)=-16a\pi^2-2b\pi i,\ \lambda_2(a,b)=-16a\pi^2,$$
$$\lambda_3(a,b)=-16a\pi^2+2b\pi i,\ \lambda_4(a,b)=-28a\pi^2,$$
$$\lambda_{4n+1}(a,b)=-4a(2n+1)^2\pi^2-\frac{b^2}{4a},$$
$$\lambda_{4n+2}(a,b)=-16a(n+1)^2\pi^2-2b(n+1)\pi i,$$
$$\lambda_{4n+3}(a,b)=-16a(n+1)^2\pi^2+2b(n+1)\pi i,$$
$$\lambda_{4n+4}(a,b)=-16a(n+1)^2\pi^2-\frac{b^2}{4a},\ n\in\mathbb{N}.$$

It is easy to see that for each point $(\widehat{a},\widehat{b})\in\Gamma_2$,

$$\lim_{\Gamma_1\ni(a,b)\to(\widehat{a},\widehat{b})}\lambda_1(a,b)=\lambda_2(\widehat{a},\widehat{b}),\quad \lim_{\Gamma_1\ni(a,b)\to(\widehat{a},\widehat{b})}\lambda_2(a,b)=\lambda_1(\widehat{a},\widehat{b}),$$
$$\lim_{\Gamma_1\ni(a,b)\to(\widehat{a},\widehat{b})}\lambda_m(a,b)=\lambda_m(\widehat{a},\widehat{b}),\ m\geq 3.$$

5. Continuity Dependence of the Spectral Gap on the Coefficients a, b and Distributions ν_0, ν_1

Now we turn to study the continuity dependence of the spectral gap on the coefficients a, b and distributions ν_0, ν_1 (see Theorem 1.2). We believe that Proposition 5.1, which will be stated first and is needed for the proof of Theorem 1.2, is of independent interest.

Proposition 5.1. (1) Zero is an algebraically simple eigenvalue of (1.1);
(2) All the nonzero eigenvalues of (1.1) have strictly positive real part.

Proof. (1) In view of Remark 2.9, it is easy to verify that zero is an eigenvalue of the problem (1.1), with the corresponding eigenfunction being any non-zero constant. We just need to calculate its algebraic multiplicity. Denote

$$y'_{1,\lambda}(x,\lambda):=\frac{\mathrm{d}y_1(x,\lambda)}{\mathrm{d}\lambda}\text{ and }y'_{2,\lambda}(x,\lambda):=\frac{\mathrm{d}y_2(x,\lambda)}{\mathrm{d}\lambda}.$$

Then from Lemma 2.6 one has

$$\begin{aligned}\Delta'(\lambda) \;=\; & \int_0^1 y'_{1,\lambda}(x,\lambda)\mathrm{d}\nu_0\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_1+\int_0^1 y_1(x,\lambda)\mathrm{d}\nu_0\int_0^1 y'_{2,\lambda}(x,\lambda)\mathrm{d}\nu_1\\
&-\int_0^1 y'_{1,\lambda}(x,\lambda)\mathrm{d}\nu_1\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0-\int_0^1 y_1(x,\lambda)\mathrm{d}\nu_1\int_0^1 y'_{2,\lambda}(x,\lambda)\mathrm{d}\nu_0\\
&+y'_{1,\lambda}(1,\lambda)\int_0^1 y_2(x,\lambda)\mathrm{d}\nu_0+y_1(1,\lambda)\int_0^1 y'_{2,\lambda}(x,\lambda)\mathrm{d}\nu_0\\
&-y'_{2,\lambda}(1,\lambda)\int_0^1 y_1(x,\lambda)\mathrm{d}\nu_0-y_2(1,\lambda)\int_0^1 y'_{1,\lambda}(x,\lambda)\mathrm{d}\nu_0\\
&-\int_0^1 y'_{2,\lambda}(x,\lambda)\mathrm{d}\nu_1+y'_{2,\lambda}(1,\lambda).\end{aligned}$$

Now we aim to show that $\Delta'(0)<0$. In fact, direct calculation yields $y_1(x,0)=1$ and

$$y_2(x,0)=\int_0^x\exp\left(-\int_0^t\frac{b(s)}{a(s)}\mathrm{d}s\right)\mathrm{d}t.$$

Then from Remark 2.2, it follows that

$$\begin{aligned} y_{1,\lambda}'(x,0) &= \int_0^x \frac{y_2(x,0)y_1(t,0)-y_1(x,0)y_2(t,0)}{a(t)\exp\left(-\int_0^t \frac{b(s)}{a(s)}\mathrm{d}s\right)}\mathrm{d}t \\ &= \int_0^x \left[y_2(x,0)-y_2(t,0)\right]\frac{\exp\left(\int_0^t \frac{b(s)}{a(s)}\mathrm{d}s\right)}{a(t)}\mathrm{d}t. \end{aligned}$$

Hence

$$\begin{aligned} \Delta'(0) = &\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_0 \int_0^1 y_2(x,0)\mathrm{d}\nu_1 - \int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_1 \int_0^1 y_2(x,0)\mathrm{d}\nu_0 \\ &+y_{1,\lambda}'(1,0)\int_0^1 y_2(x,0)\mathrm{d}\nu_0 - y_2(1,0)\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_0. \end{aligned} \tag{5.1}$$

Therefore, in order to prove $\Delta'(0)<0$, it is sufficient to prove

$$\frac{\int_0^1 y_2(x,0)\mathrm{d}\nu_0(x)}{\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_0(x)} < \frac{y_2(1,0)-\int_0^1 y_2(x,0)\mathrm{d}\nu_1(x)}{y_{1,\lambda}'(1,0)-\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_1(x)}. \tag{5.2}$$

Note that

$$\begin{aligned} &y_{1,\lambda}'(1,0)\int_0^1 y_2(x,0)\mathrm{d}\nu_0(x) - y_2(1,0)\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_0(x) \\ = &\int_0^1 y_2(x,0)\int_0^1 \left[y_2(1,0)-y_2(t,0)\right]\frac{\exp\left(\int_0^t \frac{b(s)}{a(s)}\mathrm{d}s\right)}{a(t)}\mathrm{d}t\mathrm{d}\nu_0(x) \\ &-\int_0^1 y_2(1,0)\int_0^x \left[y_2(x,0)-y_2(t,0)\right]\frac{\exp\left(\int_0^t \frac{b(s)}{a(s)}\mathrm{d}s\right)}{a(t)}\mathrm{d}t\mathrm{d}\nu_0(x) \\ = &\int_0^1\int_x^1 \left[y_2(1,0)-y_2(t,0)\right]y_2(x,0)\frac{\exp\left(\int_0^t \frac{b(s)}{a(s)}\mathrm{d}s\right)}{a(t)}\mathrm{d}t\mathrm{d}\nu_0(x) \\ &+\int_0^1\int_0^x \left[y_2(1,0)-y_2(x,0)\right]y_2(t,0)\frac{\exp\left(\int_0^t \frac{b(s)}{a(s)}\mathrm{d}s\right)}{a(t)}\mathrm{d}t\mathrm{d}\nu_0(x) \\ < &\ 0, \end{aligned} \tag{5.3}$$

and similarly,

$$y_{1,\lambda}'(1,0)\int_0^1 y_2(x,0)\mathrm{d}\nu_1(x) - y_2(1,0)\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_1(x) < 0. \tag{5.4}$$

Therefore, (5.3) and (5.4) yield that

$$\frac{\int_0^1 y_2(x,0)\mathrm{d}\nu_0(x)}{\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_0(x)} < \frac{y_2(1,0)}{y_{1,\lambda}'(1,0)} < \frac{y_2(1,0)-\int_0^1 y_2(x,0)\mathrm{d}\nu_1(x)}{y_{1,\lambda}'(1,0)-\int_0^1 y_{1,\lambda}'(x,0)\mathrm{d}\nu_1(x)}.$$

This proves (5.2) and thus $\Delta'(0)<0$ which implies that zero is an algebraically simple eigenvalue of the problem (1.1).

(2) Let μ be an eigenvalue of the problem (1.1) , and denote the corresponding eigenfunction by y.

Step 1: We first show that there exists a point $x_0 \in (0,1)$ such that $|y(x_0)|^2 = M := \max_{x\in[0,1]} |y(x)|^2$. Suppose $|y(0)|^2 = M$. Using the boundary conditions of (1.1) and the Cauchy-Schwartz inequality, we thus deduce that for each $x \in [0,1]$,

$$\begin{aligned}|y(x)|^2 &\le |y(0)|^2 = \left|\int_0^1 y(x)\mathrm{d}\nu_0(x)\right|^2 \\ &\le \left(\int_0^1 |y(x)|\,\mathrm{d}\nu_0(x)\right)^2 \le \int_0^1 |y(x)|^2\,\mathrm{d}\nu_0(x).\end{aligned} \tag{5.5}$$

Thus

$$\int_0^1 |y(x)|^2\,\mathrm{d}\nu_0(x) = \left(\int_0^1 |y(x)|\,\mathrm{d}\nu_0(x)\right)^2. \tag{5.6}$$

This yields that

$$\int_0^1 \left(|y(x)| - \int_0^1 |y(x)|\,\mathrm{d}\nu_0(x)\right)^2 \mathrm{d}\nu_0(x) = 0,$$

and hence $|y(x)| \equiv \int_0^1 |y(x)|\,\mathrm{d}\nu_0(x)$ ν_0-a.e.

Then it follows from (5.5) that $|y(x)|^2 \equiv |y(0)|^2 = M$ ν_0-a.e. Suppose $|y(1)|^2 = M$, then similar proof yields that $|y(x)|^2 \equiv |y(1)|^2 = M$ ν_1-a.e. Therefore, we conclude that there exists a point $x_0 \in (0,1)$ such that $|y(x_0)|^2 = M$.

Step 2: Note that

$$\begin{aligned}l|y|^2 &= a\left(|y|^2\right)'' + b\left(|y|^2\right)' = a\,(y\overline{y})'' + b\,(y\overline{y})' \\ &= \overline{y}ly + yl\overline{y} + 2a|y'|^2 \le \overline{y}ly + yl\overline{y} = 2\mathrm{Re}\mu|y|^2.\end{aligned} \tag{5.7}$$

Assume $\mathrm{Re}\mu \le 0$, then we have $l|y|^2 \le 0$. Thus based on the conclusion in Step 1, we deduce from maximum principle ([22, Section 6.4, Theorem 3]) that $|y(x)|^2 \equiv c$ on $[0,1]$ for some constant c, and hence $l|y|^2 \equiv 0$. Therefore, it follows from (5.7) that $\mathrm{Re}\mu = 0$ and $|y'(x)|^2 \equiv 0$. Then it is easy to see that $y(x) \equiv c_1$ on $[0,1]$ for some constant c_1. Thus $\mu = 0$. Now we can conclude that zero is the only eigenvalue with non-positive real part. This completes the proof. ☐

Remark 5.2. In fact, the second part of Proposition 5.1 was given in [2] by a different method.

Now Theorem 1.2 announced in the introduction follows directly from Proposition 5.1 and Theorem 1.1. In fact, in view of the meaning of λ_1, one easily deduces from Proposition 5.1 that $\lambda_1(\omega)$ $(\lambda_1(\omega_n))$ is always the nonzero eigenvalue of the problem ω (ω_n) with the minimal real part. This together with the statement (2) of Theorem 1.1 clearly implies the assertion of Theorem 1.2.

We conclude this section by a concrete example.

Example 5.3. Denote the spectral gap of the problem (4.25) by $\gamma_1(\delta_{\frac{1}{2}})$, i.e.,

$$\gamma_1(\delta_{\frac{1}{2}}) := \inf\left\{\mathrm{Re}\lambda \,|\, \lambda \text{ is an eigenvalue of the problem (4.25) and } \lambda \ne 0\right\}.$$

It follows from Example 4.8 that

$$\gamma_1(\delta_{\frac{1}{2}}) = \begin{cases} -4a\pi^2 - \frac{b^2}{4a}, & \text{when } |b| \le -4\sqrt{3}a\pi, \\ -16a\pi^2, & \text{when } |b| > -4\sqrt{3}a\pi. \end{cases} \tag{5.8}$$

Then it is obvious that as a function of $(a, b) \in (-\infty, 0) \times \mathbb{R}$, $\gamma_1(\delta_{\frac{1}{2}})$ is continuous on the region $(-\infty, 0) \times \mathbb{R}$, which is consistent with the statement of Theorem 1.2. Note that (5.8) is already given in [14] and [13] by different approaches.

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