

INFINITE TIME BLOW-UP FOR THE FRACTIONAL HEAT EQUATION WITH CRITICAL EXPONENT

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ABSTRACT. We consider positive solutions for the fractional heat equation with critical exponent

$$\begin{cases} u_t = -(-\Delta)^s u + u^{\frac{n+2s}{n-2s}} & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } (\mathbb{R}^n \setminus \Omega) \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^n , $n > 4s$, $s \in (0, 1)$, $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ and u_0 is a positive smooth initial datum with $u_0|_{\mathbb{R}^n \setminus \Omega} = 0$. We prove the existence of u_0 such that the solution blows up precisely at prescribed distinct points q_1, \dots, q_k in Ω as $t \rightarrow +\infty$. The main ingredient of the proofs is a new inner-outer gluing scheme for the fractional parabolic problems.

1. INTRODUCTION

Let Ω be a smooth bounded domain in \mathbb{R}^n , $n \geq 1$. We consider the fractional heat equation with critical exponent

$$\begin{cases} u_t = -(-\Delta)^s u + u^{\frac{n+2s}{n-2s}} & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } (\mathbb{R}^n \setminus \Omega) \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \quad (1.1)$$

for a function $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ and a smooth, positive initial datum u_0 satisfying $u_0|_{\mathbb{R}^n \setminus \Omega} = 0$, $s \in (0, 1)$. Here, for any point $x \in \mathbb{R}^n$, the fractional Laplace operator $(-\Delta)^s u(x)$ is defined as

$$(-\Delta)^s u(x) := C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy$$

with a suitable positive normalizing constant $C(n, s)$. We refer to [27] for an introduction to the fractional Laplace operator and to the appendix of [18] for a heuristic physical motivation in nonlocal quantum mechanics of the fractional operator considered here.

Parabolic problems like (1.1) and related ones have attracted much attention in recent years, for example, [2], [3], [8], [9], [10], [11], [12], [15], [28], [29], [41], [42] and the references therein. As in the case of $s = 1$, problem (1.1) is the formal negative L^2 -gradient flow of the functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{n-2s}{2n} \int_{\Omega} |u|^{\frac{2n}{n-2s}} dx$$

in

$$H_0^s(\Omega) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 dx < +\infty \text{ and } u = 0 \right. \\ \left. \text{almost everywhere in } \mathbb{R}^n \setminus \Omega \right\},$$

i.e., $\frac{d}{dt} E(u(\cdot, t)) = - \int_{\mathbb{R}^n} |u_t|^2 dx$. If the function $u(x, t)$ is independent of t , (1.1) is a semilinear elliptic problem with fractional Laplacian, which has been studied extensively, for instance, in [16] and [40].

When $s = 1$, problem (1.1) is the classical parabolic equation with critical exponent

$$\begin{cases} u_t = \Delta u + u^{\frac{n+2}{n-2}} & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in } \Omega. \end{cases} \quad (1.2)$$

Many authors are interested in the blow-up phenomenon of (1.2), for example, [17], [25], [26], [31], [36], [37], [38], [39]. In [17], Cortazar, del Pino and Musso obtained the following result. Suppose $n > 4$, let $\hat{G}(x, y)$ be the Green's function of $-\Delta$ in Ω with Dirichlet boundary value and $\hat{H}(x, y)$ be its regular part. Given k distinct points q_1, \dots, q_k in Ω such that the matrix

$$\hat{\mathcal{G}}(q) = \begin{bmatrix} \hat{H}(q_1, q_1) & -\hat{G}(q_1, q_2) & \cdots & -\hat{G}(q_1, q_k) \\ -\hat{G}(q_2, q_1) & \hat{H}(q_2, q_2) & \cdots & -\hat{G}(q_2, q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{G}(q_k, q_1) & -\hat{G}(q_k, q_2) & \cdots & \hat{H}(q_k, q_k) \end{bmatrix}$$

is positive definite, they proved the existence of an initial datum u_0 and smooth parameter functions $\xi_j(t) \rightarrow q_j$, $0 < \mu_j(t) \rightarrow 0$, as $t \rightarrow +\infty$, $j = 1, \dots, k$, such that there exists an infinite time blow-up solution u_q of (1.2) which has the approximate form

$$u_q \approx \sum_{j=1}^k \alpha_n \left(\frac{\mu_j(t)}{\mu_j^2(t) + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}$$

with $\mu_j(t) = \beta_j t^{-\frac{1}{n-4}} (1 + o(1))$ for certain positive constants β_j . The aim of this paper is to show that this phenomenon also occurs in problem (1.1). Our starting point is the positive entire solutions of the equation

$$-(-\Delta)^s U + U^{\frac{n+2s}{n-2s}} = 0 \text{ in } \mathbb{R}^n,$$

which are given by the bubbles

$$U_{\mu, \xi}(x) = \mu^{-\frac{n-2s}{2}} U_0 \left(\frac{x - \xi}{\mu} \right) = \alpha_{n,s} \left(\frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{n-2s}{2}}, \quad (1.3)$$

where

$$U_0(y) = \alpha_{n,s} \left(\frac{1}{1 + |y|^2} \right)^{\frac{n-2s}{2}}$$

and $\alpha_{n,s}$ is a constant depending only on n and s , see, [14] and [35]. Let $G(x, y)$ be the Green's function for the following nonlocal problem

$$\begin{cases} (-\Delta)^s G(x, y) = c(n, s)\delta(x - y) & \text{in } \Omega, \\ G(\cdot, y) = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $\delta(x)$ denotes the Dirac measure at the origin and $c(n, s)$ satisfies

$$(-\Delta)^s \Gamma(x) = c(n, s)\delta(x), \quad \Gamma(x) = \frac{\alpha_{n,s}}{|x|^{n-2s}}.$$

The regular part of $G(x, y)$ is denoted by $H(x, y)$, namely $H(x, y)$ solves the following problem

$$\begin{cases} (-\Delta)^s H(x, y) = 0 & \text{in } \Omega, \\ H(\cdot, y) = \Gamma(\cdot - y) & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Let $q = (q_1, \dots, q_k)$ be the collection of k distinct points in Ω and define

$$\mathcal{G}(q) := \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_2, q_1) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -G(q_k, q_1) & -G(q_k, q_2) & \cdots & H(q_k, q_k) \end{bmatrix}. \quad (1.4)$$

Our main result is stated as follows.

Theorem 1.1. *Assume $n > 4s$, $s \in (0, 1)$ and q_1, \dots, q_k are distinct points in Ω such that the matrix (1.4) is positive definite. Then there exist u_0 and smooth parameter functions $\xi_j(t) \rightarrow q_j$, $0 < \mu_j(t) \rightarrow 0$, as $t \rightarrow +\infty$, $j = 1, \dots, k$, such that there exists solution u_q to problem (1.1) with the form*

$$u_q = \sum_{j=1}^k \alpha_{n,s} \left(\frac{\mu_j(t)}{\mu_j^2(t) + |x - \xi_j(t)|^2} \right)^{\frac{n-2s}{2}} - \mu_j^{\frac{n-2s}{2}}(t)H(x, q_j) + \mu_j^{\frac{n-2s}{2}}(t)\varphi(x, t),$$

where $\varphi(x, t)$ is bounded satisfying $\varphi(x, t) \rightarrow 0$ as $t \rightarrow +\infty$, uniformly away from q_j . Furthermore, there exists a submanifold \mathcal{M} with codimension k in $X := \{u \in C^1(\mathbb{R}^n) : u|_{\mathbb{R}^n \setminus \Omega} = 0\}$ containing $u_q(x, 0)$ such that, if u_0 is a small perturbation of $u_q(x, 0)$ in \mathcal{M} , then the solution $u(x, t)$ of (1.1) still has the form

$$u(x, t) = \sum_{j=1}^k \alpha_{n,s} \left(\frac{\tilde{\mu}_j(t)}{\tilde{\mu}_j^2(t) + |x - \tilde{\xi}_j(t)|^2} \right)^{\frac{n-2s}{2}} - \tilde{\mu}_j^{\frac{n-2s}{2}}(t)H(x, \tilde{q}_j) + \tilde{\mu}_j^{\frac{n-2s}{2}}(t)\tilde{\varphi}(x, t),$$

where the point \tilde{q}_j is close to q_j for $j = 1, \dots, k$.

In order to prove this theorem, we shall develop a new *inner-outer gluing scheme* for fractional parabolic problems. It is well-known that gluing methods have been proven to be very useful in singular perturbation elliptic problems, for example, [22], [23], [24]. This method has also been applied in various parabolic flows recently, such as the infinite time blow-up for critical nonlinear heat equation [17], [26] and half-harmonic map flow [43], the singularity formation for two dimensional harmonic map flow [20], finite time blow-up for critical nonlinear heat equation [25], type II ancient solution for Yamabe flow [21].

When dealing with parabolic problems, a crucial step in the scheme is to find a solution of the linearized parabolic equation around the bubble with sufficiently fast decay. However, it seems that the local argument in [17] for the classical critical heat

equation does not work in the fractional case. Inspired by Lemma 4.5 of [20] and the linear theory of [43], we will use a blow-up argument based on the nondegeneracy of bubbles and a removable singularity property for the corresponding limit equations. (See Section 5 below.)

As mentioned, the case of fractional parabolic problems is much more intricate. For the semilinear equation, Sugitani [44] proved non-existence below the Fujita exponent $p_* = 1 + \frac{2s}{n}$. The case of global existence above the exponent remains open since all the known techniques fail in this case. Related to a similar question, the paper [33] provides an optimal initial trace theory (see also [4] for the case of the homogeneous fractional heat equation). As far as blow-up is concerned, a theory in the spirit of the one developed for instance by Giga and Kohn [32], is missing. A crucial step in these approaches is to exhibit a monotone quantity. For the operator $\partial_t + (-\Delta)^s$, such a quantity is missing. On the other hand, as far as nonlocal operators are concerned, the fractional power of the heat equation, i.e. $(\partial_t - \Delta)^s$, on the other hand exhibits monotonicity (see for instance [?]). Notice that the latter operator has the same stationary solutions as the former one.

The proof of Theorem 1.1 is rather long. We outline the proof and point out key arguments here. To explain the idea, we assume that $k = 1$ in the rest of this section.

Step 1. Construction of approximation. Our aim is to find a solution $u(x, t)$ in the following approximate form

$$u(x, t) \approx U_{\mu(t), \xi(t)}(x)$$

with $\xi(t) \rightarrow q$, $\mu(t) \rightarrow 0$ as $t \rightarrow \infty$ and $U_{\mu(t), \xi(t)}(x)$ is defined in (1.3). Denote the error operator as

$$S(u) := -u_t - (-\Delta)^s u + u^p,$$

where $p = \frac{n+2s}{n-2s}$. Then the error of $U_{\mu(t), \xi(t)}(x)$ is

$$S(U_{\mu(t), \xi(t)}) = \mu^{-\frac{n-2s}{2}-1} \dot{\mu} Z_{n+1}(y) + \mu^{-\frac{n-2s}{2}-1} \dot{\xi} \cdot \nabla U(y).$$

Here $y = \frac{x-\xi(t)}{\mu(t)}$. It turns out that the terms $\mu^{-\frac{n-2s}{2}-1} \dot{\mu} Z_{n+1}(y)$ and $\mu^{-\frac{n-2s}{2}-1} \dot{\xi} \cdot \nabla U(y)$ do not have enough decay to perform the gluing method we shall use. (For $s = 1$, this is enough.) So we add a nonlocal term $\Phi^*(x, t) = \Phi^0(x, t) + \Phi^1(x, t)$ to cancel them out at main order. Since $u = 0$ in $\mathbb{R}^n \setminus \Omega$, a better approximation than $U_{\mu(t), \xi(t)}(x)$ is

$$u_{\mu, \xi}(x, t) = U_{\mu, \xi}(x) + \mu^{\frac{n-2s}{2}} \Phi^*(x, t) - \mu^{\frac{n-2s}{2}} H(x, q).$$

The error of $u_{\mu, \xi}$ can be computed as

$$\mu^{\frac{n+2s}{2}} S(u_{\mu, \xi}) \approx \mu E_0 + \mu E_1$$

with

$$\begin{aligned} E_0 &= pU(y)^{p-1} (-\mu^{n-2s-1} H(q, q)) + pU(y)^{p-1} \mu^{n-2s-1} \Phi^0(q, t) \\ &\quad + \mu^{2s-2} \dot{\mu} \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \end{aligned}$$

and

$$E_1 = pU(y)^{p-1} (-\mu^{n-2s}\nabla H(q, q)) \cdot y + pU(y)^{p-1}\mu^{n-2s-1}\Phi^1(q, t) \\ + \alpha_{n,s}(n-2s)\mu^{2s-2} \frac{\dot{\xi} \cdot y}{(1+|y|^2)^{\frac{n-2s}{2}+1}}.$$

We shall look for solutions with the form

$$u(x, t) = u_{\mu,\xi} + \tilde{\phi}(x, t),$$

where

$$\tilde{\phi}(x, t) = \mu^{-\frac{n-2s}{2}} \phi\left(\frac{x-\xi}{\mu}, t\right).$$

By $S(u) = S(u_{\mu,\xi} + \tilde{\phi}(x, t)) = 0$, the equation for $\phi(y, t)$ is

$$-(-\Delta)_y^s \phi + pU(y)^{p-1} \phi + \mu^{\frac{n+2s}{2}} S(u_{\mu,\xi}) + A[\phi] = 0$$

for a small term $A[\phi]$. Note that in the expansion of $\mu^{\frac{n+2s}{2}} S(u_{\mu,\xi})$, the largest term is μE_0 , so $\phi(y, t)$ should equal a solution $\phi_0(y, t)$ of the following elliptic type equation at main order

$$-(-\Delta)_y^s \phi_0 + pU(y)^{p-1} \phi_0 = -\mu_0 E_0 \text{ in } \mathbb{R}^n, \quad \phi_0(y, t) \rightarrow 0 \text{ as } |y| \rightarrow \infty. \quad (1.5)$$

Equation (1.5) is a special case of

$$L_0[\psi] := -(-\Delta)_y^s \psi + pU(y)^{p-1} \psi = h(y) \text{ in } \mathbb{R}^n, \quad \psi(y) \rightarrow 0 \text{ as } |y| \rightarrow \infty. \quad (1.6)$$

It is well known that every bounded solution of $L_0[\psi] = 0$ in \mathbb{R}^n is the linear combination of the functions

$$Z_1, \dots, Z_{n+1}$$

where

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \dots, n, \quad Z_{n+1}(y) := \frac{n-2s}{2}U(y) + y \cdot \nabla U(y).$$

The above non-degeneracy result can be found in [19]. Furthermore, problem (1.6) is solvable for $h(y) = O(|y|^{-m})$, $m > 2s$, if it holds that

$$\int_{\mathbb{R}^n} h(y) Z_i(y) dy = 0 \quad \text{for } i = 1, \dots, n+1$$

By choosing $\bar{\mu}_0 = b\mu_0(t)$ for suitable positive constant b and $\xi_0 = q$, the solvability conditions

$$\int_{\mathbb{R}^n} \mu_0 E_0(y, t) Z_i(y) dy = 0, \quad i = 1, \dots, n+1 \quad (1.7)$$

can be achieved at main order. Here $\mu_0(t) = c_{n,s} t^{-\frac{1}{n-4s}}$ for some constant $c_{n,s}$. Under the solvability condition (1.7), (1.5) has a solution $\Phi(y, t)$, which leads to the following corrected approximation

$$u_{\mu,\xi}^*(x, t) = u_{\mu,\xi}(x, t) + \tilde{\Phi}(x, t),$$

where

$$\tilde{\Phi}(x, t) = \mu^{-\frac{n-2s}{2}} \Phi\left(\frac{x-\xi}{\mu}, t\right)$$

and $\mu(t) = b\mu_0(t) + \lambda(t)$. Finally, we use the ansatz

$$u = u_{\mu,\xi}^* + \tilde{\phi}.$$

We shall show the details and the general case $k \geq 1$ in Section 2.

Step 2. The inner-outer gluing procedure. Denote

$$\tilde{\phi}(x, t) = \psi(x, t) + \phi^{in}(x, t), \quad \text{where} \quad \phi^{in}(x, t) := \eta_R \tilde{\phi}(x, t)$$

with

$$\tilde{\phi}(x, t) := (b\mu_0)^{-\frac{n-2s}{2}} \phi\left(\frac{x-\xi}{b\mu_0}, t\right)$$

and

$$\eta_R(x, t) = \eta\left(\frac{|x-\xi|}{R\mu_0}\right).$$

The cut-off function $\eta(\tau)$ satisfies $\eta(\tau) = 1$ for $0 \leq \tau < 1$ and $\eta(\tau) = 0$ for $\tau > 2$. The number R is independent of t and fixed sufficiently large. In terms of $\tilde{\phi}$, problem (1.1) can be expressed as

$$\begin{cases} \partial_t \tilde{\phi} = -(-\Delta)^s \phi + p(u_{\mu, \xi}^*)^{p-1} \tilde{\phi} + \tilde{N}(\tilde{\phi}) + S(\mu_{\mu, \xi}^*), & \text{in } \Omega \times (t_0, \infty), \\ \tilde{\phi} = -u_{\mu, \xi}^*, & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty). \end{cases} \quad (1.8)$$

Let

$$V_{\mu, \xi} = p \left((u_{\mu, \xi}^*)^{p-1} - \left[\mu^{-\frac{n-2s}{2}} U\left(\frac{x-\xi}{\mu}\right) \right]^{p-1} \right) \eta_R + p(1 - \eta_R)(u_{\mu, \xi}^*)^{p-1}.$$

Then $\tilde{\phi}$ solves problem (1.8) if ψ and ϕ satisfy the following two coupled equations respectively,

$$\begin{cases} \partial_t \psi = -(-\Delta)^s \psi + V_{\mu, \xi} \psi + \tilde{\phi}(-(-\Delta)^s) \eta_R + \dots, & \text{in } \Omega \times (t_0, \infty), \\ \psi = -u_{\mu, \xi}^*, & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty) \end{cases} \quad (1.9)$$

and

$$\begin{aligned} \mu_0^{2s} \partial_t \phi &= -(-\Delta)_y^s \phi + pU^{p-1}(y)\phi \\ &+ \left\{ p\mu_0^{\frac{n-2s}{2}} \frac{\mu_0^{2s}}{\mu^{2s}} U^{p-1}\left(\frac{\mu_0}{\mu}y\right) \psi(\xi + \mu_0 y, t) + \dots \right\} \chi_{B_{2R}(0)}(y), y \in \mathbb{R}^n. \end{aligned} \quad (1.10)$$

(1.9) is the so-called *outer problem* and (1.10) is the *inner problem*. Note that the inner problem is solved in the *whole space* with error supported in $B_{2R}(0)$. See Section 3 for details.

Step 3. The outer problem. For a fixed $a > 2s$, we solve the outer problem (1.9) for ψ under the initial condition $\psi(\cdot, t_0) = \psi_0$ in \mathbb{R}^n . Suppose

$$(1 + |y|)|\nabla_y \phi(y, t)| \chi_{B_{2R}(0)}(y) + |\phi(y, t)| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2s+\sigma}(t)}{1 + |y|^a} \quad (1.11)$$

holds for a small constant $\sigma > 0$ and small $\varepsilon > 0$. Using the super-sub solution method, we solve (1.9) and obtain the existence of a unique solution $\psi = \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$ satisfying

$$|\psi(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \frac{\mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1 + |y|^{a-2s}} + e^{-\delta(t-t_0)} \|\psi_0\|_{L^\infty(\mathbb{R}^n)}$$

and

$$[\psi(x, t)]_{\eta, B_{\mu R}(\xi)} \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \frac{\mu^{-\eta} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1 + |y|^{a-2s+\eta}} \text{ for } |y| \leq 2R,$$

where $y = \frac{x-\xi}{\mu_0}$. This is the content of Section 4.

After substituting $\psi = \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$ into the inner problem (1.10) and using the change of variables $\frac{dt}{d\tau} = \mu_0^{2s}(t)$, the full problem is reduced to the solvability of the following nonlinear nonlocal equation

$$\begin{cases} \partial_\tau \phi = -(-\Delta)_y^s \phi + pU^{p-1}(y)\phi + H[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)), & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \phi(y, \tau_0) = e_0 Z_0(y), & y \in \mathbb{R}^n, \end{cases} \quad (1.12)$$

for some constant e_0 , and Z_0 is the bounded eigenfunction corresponding to the only negative eigenvalue λ_0 to the following eigenvalue problem

$$-(-\Delta)^s \phi + pU^{p-1}\phi + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).$$

Step 4. Linear theory for (1.12). To solve the problem (1.12), we first consider the following linear parabolic problem

$$\begin{cases} \partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau), & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \phi(y, \tau_0) = e_0 Z_0(y), & y \in \mathbb{R}^n. \end{cases} \quad (1.13)$$

Assuming $h(\cdot, \tau)$ is supported in $B_{2R}(0)$ for any $\tau \geq \tau_0$, $\|h\|_{2s+a, \nu, \eta} < +\infty$ and

$$\int_{B_{2R}(0)} h(y, \tau) Z_j(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \infty), \quad j = 1, \dots, n+1,$$

we prove the existence of a fast-decaying solution $\phi = \phi[h](y, \tau)$ and $e_0 = e_0[h](\tau)$ ($\tau \in (\tau_0, +\infty)$, $y \in \mathbb{R}^n$) solving problem (1.13). In addition, the following estimates hold,

$$\begin{aligned} (1 + |y|) |\nabla_y \phi(y, \tau)|_{\chi_{B_{2R}(0)}}(y) + |\phi(y, \tau)| \\ \lesssim \tau^{-\nu} (1 + |y|)^{-a} \|h\|_{2s+a, \nu, \eta}, \quad \tau \in (\tau_0, +\infty), y \in \mathbb{R}^n \end{aligned}$$

and

$$|e_0[h]| \lesssim \|h\|_{2s+a, \nu, \eta}.$$

It seems that the linear theory in [17] does not work in the fractional case, instead, we will use the blow up argument similar to [20]. Here we need the technical assumption $a > 2s$ to ensure the integrability. This is the reason why we add two nonlocal terms in **Step 1** and there is a term $\frac{1}{R^{a-2s}}$ in the estimation of ψ , see Section 5.1.

Step 5. The solvability condition for (1.12). From **Step 4**, we see that problem (1.12) is solvable for functions ϕ satisfying (1.11), provided ξ and λ are chosen such that

$$\int_{B_{2R}} H[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)) Z_l(y) dy = 0, \quad \text{for all } \tau \geq \tau_0, l = 1, 2, \dots, n+1.$$

By the orthogonality conditions above, our original problem is reduced to a nonlinear nonlocal system of ODEs for λ and ξ , which is achieved in Section 5.2.

Step 6. The inner problem: gluing. We finally solve the nonlinear nonlocal problem (1.12) based on the linear theory for (1.13) and the Contraction Mapping Theorem. See Section 6 for details.

2. CONSTRUCTION OF THE APPROXIMATION

2.1. Setting up the problem. Let $t_0 > 0$. We consider the following evolution problem

$$\begin{cases} u_t = -(-\Delta)^s u + u^{\frac{n+2s}{n-2s}} & \text{in } \Omega \times (t_0, \infty), \\ u = 0 & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \end{cases} \quad (2.1)$$

which provides a solution $u(x, t) = u(x, t - t_0)$ to (1.1). Given k points $q_1, \dots, q_k \in \mathbb{R}^n$, our aim is to find a solution $u(x, t)$ of (2.1) in the following approximate form

$$u(x, t) \approx \sum_{j=1}^k U_{\mu_j(t), \xi_j(t)}(x)$$

with $\xi_j(t) \rightarrow q_j$, $\mu_j(t) \rightarrow 0$ as $t \rightarrow \infty$ for all $j = 1, \dots, k$ and $U_{\mu_j(t), \xi_j(t)}(x)$ is defined in (1.3). Denote the error operator as

$$S(u) := -u_t - (-\Delta)^s u + u^p,$$

where $p = \frac{n+2s}{n-2s}$. Then the error of $U_{\mu_j(t), \xi_j(t)}(x)$ is

$$\begin{aligned} S(U_{\mu_j(t), \xi_j(t)}) &= -\frac{\partial}{\partial t} U_{\mu_j, \xi_j}(x) = \mu_j^{-\frac{n-2s}{2}} \left(\frac{\dot{\mu}_j}{\mu_j} Z_{n+1}(y_j) + \frac{\dot{\xi}_j}{\mu_j} \cdot \nabla U(y_j) \right) \\ &= \mu_j^{-\frac{n-2s}{2}-1} \dot{\mu}_j Z_{n+1}(y_j) + \mu_j^{-\frac{n-2s}{2}-1} \dot{\xi}_j \cdot \nabla U(y_j). \end{aligned}$$

Here $y_j = \frac{x - \xi_j(t)}{\mu_j(t)}$. It turns out that the terms $\mu_j^{-\frac{n-2s}{2}-1} \dot{\mu}_j Z_{n+1}(y_j)$ and $\mu_j^{-\frac{n-2s}{2}-1} \dot{\xi}_j \cdot \nabla U(y_j)$ do not have enough decay to perform the gluing method, so we add nonlocal terms to cancel them out at main order. Note that the main order of

$$Z_{n+1}(y) = \frac{n-2s}{2} \alpha_{n,s} \frac{1 - |y|^2}{(1 + |y|^2)^{\frac{n-2s}{2}+1}}$$

is

$$-\frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y|^2)^{\frac{n-2s}{2}}}.$$

Therefore, we consider the equation

$$-\varphi_t - (-\Delta)^s \varphi - \frac{n-2s}{2} \alpha_{n,s} \frac{\dot{\mu}_j}{\mu_j} \frac{\mu_j^{-(n-2s)}}{\left(1 + \left|\frac{x - \xi_j}{\mu_j}\right|^2\right)^{\frac{n-2s}{2}}} = 0 \text{ in } \mathbb{R}^n \times (t_0, +\infty). \quad (2.2)$$

Then

$$\Phi_j^0(x, t) = - \int_{t_0}^t \int_{\mathbb{R}^n} p(t - \tilde{s}, x - y) \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \frac{\mu_j^{-(n-2s)}(\tilde{s})}{\left(1 + \left|\frac{y - \xi_j(\tilde{s})}{\mu_j(\tilde{s})}\right|^2\right)^{\frac{n-2s}{2}}} dy d\tilde{s}$$

is a bounded solution for (2.2). Here the function $p(t, x)$ is the heat kernel for the fractional heat operator $-\frac{\partial}{\partial t} - (-\Delta)^s$, see [7] for its definition and properties. Using the super-sub solution argument (see Lemma 4.1), it is easy to see that $\Phi_j^0(x, t)$

satisfies the estimate $\Phi_j^0(x, t) \sim \frac{\dot{\mu}_j}{\mu_j} \frac{\mu_j^{-n+4s}}{1 + |y_j|^{n-4s}}$.

Similarly, for $y_j = \frac{x - \xi_j}{\mu_j}$, we consider the equation

$$-\varphi_t - (-\Delta)^s \varphi + \alpha_{n,s} (n-2s) \mu_j^{-(n-2s)-1} \frac{|y_j|^2}{(1+|y_j|^2)^{\frac{n-2s}{2}+2}} \dot{\xi}_j \cdot y_j = 0 \text{ in } \mathbb{R}^n \times (t_0, +\infty). \quad (2.3)$$

Its solution defined by

$$\Phi_j^1(x, t) = - \int_{t_0}^t \int_{\mathbb{R}^n} p(t - \tilde{s}, x - y) \mu_j^{-(n-2s)}(\tilde{s}) \frac{\dot{\xi}_j(\tilde{s}) \cdot \frac{y - \xi_j(\tilde{s})}{\mu_j(\tilde{s})}}{\mu_j(\tilde{s})} \times \frac{\left| \frac{y - \xi_j(\tilde{s})}{\mu_j(\tilde{s})} \right|^2}{\left(1 + \left| \frac{y - \xi_j(\tilde{s})}{\mu_j(\tilde{s})} \right|^2 \right)^{\frac{n-2s}{2}+2}} dy d\tilde{s}$$

satisfies the estimate $\Phi_j^1(x, t) \sim \frac{|\dot{\xi}_j|}{\mu_j} \frac{\mu_j^{-n+4s}}{1+|y_j|^{n-4s+1}}$. Define $\Phi_j^*(x, t) = \Phi_j^0(x, t) + \Phi_j^1(x, t)$. Since $u = 0$ in $\mathbb{R}^n \setminus \Omega$, a better approximation than $\sum_{j=1}^k U_{\mu_j(t), \xi_j(t)}(x)$ is

$$u_{\mu, \xi}(x, t) = \sum_{j=1}^k u_j(x, t) \text{ with } u_j(x, t) := U_{\mu_j, \xi_j}(x) + \mu_j^{\frac{n-2s}{2}} \Phi_j^*(x, t) - \mu_j^{\frac{n-2s}{2}} H(x, q_j). \quad (2.4)$$

The error of $u_{\mu, \xi}$ can be computed as

$$S(u_{\mu, \xi}) = - \sum_{i=1}^k \partial_t u_i + \left(\sum_{i=1}^k u_i \right)^p - \sum_{i=1}^k U_{\mu_i, \xi_i}^p - \sum_{i=1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t). \quad (2.5)$$

2.2. The error $S(u_{\mu, \xi})$. Near a given point q_j , we have the following estimate.

Lemma 2.1. *Consider the region $|x - q_j| \leq \frac{1}{2} \min_{i \neq l} |q_i - q_l|$ for a fixed index j , denote $x = \xi_j + \mu_j y_j$, then we have*

$$S(u_{\mu, \xi}) = \mu_j^{-\frac{n+2s}{2}} (\mu_j E_{0j} + \mu_j E_{1j} + \mathcal{R}_j)$$

with

$$\begin{aligned} E_{0j} &= pU(y_j)^{p-1} \left[-\mu_j^{n-2s-1} H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}-1} \mu_i^{\frac{n-2s}{2}} G(q_j, q_i) \right] \\ &\quad + pU(y_j)^{p-1} \mu_j^{n-2s-1} \Phi_j^0(q_j, t) \\ &\quad + \mu_j^{2s-2} \dot{\mu}_j \left(Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y_j|^2)^{\frac{n-2s}{2}}} \right), \\ E_{1j} &= pU(y_j)^{p-1} \left[-\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \cdot y_j \\ &\quad + pU(y_j)^{p-1} \mu_j^{n-2s-1} \Phi_j^1(q_j, t) + \alpha_{n,s} (n-2s) \mu_j^{2s-2} \frac{\dot{\xi}_j \cdot y_j}{(1+|y_j|^2)^{\frac{n-2s}{2}+1}} \end{aligned}$$

and

$$\mathcal{R}_j = \frac{\mu_0^{n-2s+2}g}{1+|y_j|^{4s-2}} + \frac{\mu_0^{n-2s}\vec{g}}{1+|y_j|^{4s}} \cdot (\xi_j - q_j) + \mu_0^{n+2s}f + \mu_0^{n-1} \sum_{i=1}^k \mu_i f_i + \mu_0^n \sum_{i=1}^k \dot{\xi}_i \cdot \vec{f}_i,$$

where f, f_i, \vec{f}_i are smooth, bounded functions depending on $(y, \mu_0^{-1}\mu, \xi, \mu_j y_j)$, and g, \vec{g} depend on $(y, \mu_0^{-1}\mu, \xi)$.

Proof. We write

$$u_{\mu, \xi}(x, t) = \sum_{i=1}^k \mu_i^{-\frac{n-2s}{2}} U(y_i) + \mu_i^{\frac{n-2s}{2}} \Phi_i^*(x, t) - \mu_i^{\frac{n-2s}{2}} H(x, q_i), \quad y_i = \frac{x - \xi_i}{\mu_i}$$

and

$$S(u_{\mu, \xi}) = S_1 + S_2,$$

where

$$\begin{aligned} S_1 &:= \sum_{i=1}^k \left(\mu_i^{-\frac{n-2s}{2}-1} \dot{\xi}_i \cdot \nabla U(y_i) + \mu_i^{-\frac{n-2s}{2}-1} \dot{\mu}_i Z_{n+1}(y_i) \right. \\ &\quad \left. + \frac{n-2s}{2} \mu_i^{\frac{n-2s}{2}-1} \dot{\mu}_i H(x, q_i) \right) - \sum_{i=1}^k \partial_t \left(\mu_i^{\frac{n-2s}{2}} \Phi_i^*(x, t) \right), \\ S_2 &:= \left(\sum_{i=1}^k \mu_i^{-\frac{n-2s}{2}} U(y_i) + \mu_i^{\frac{n-2s}{2}} \Phi_i^*(x, t) - \mu_i^{\frac{n-2s}{2}} H(x, q_i) \right)^p \\ &\quad - \sum_{i=1}^k \mu_i^{-\frac{n+2s}{2}} U(y_i)^p - \sum_{i=1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t). \end{aligned}$$

Let

$$S_2 = S_{21} + S_{22}$$

with

$$\begin{aligned} S_{21} &= \mu_j^{-\frac{n+2s}{2}} [(U(y_j) + \Theta_j)^p - U(y_j)^p], \\ S_{22} &= - \sum_{i \neq j} \mu_i^{-\frac{n+2s}{2}} U(y_i)^p - \sum_{i=1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t) \end{aligned}$$

and

$$\begin{aligned} \Theta_j &= - \mu_j^{n-2s} (H(x, q_j) - \Phi_j^*(x, t)) \\ &\quad + \sum_{i \neq j} \left[(\mu_j \mu_i^{-1})^{\frac{n-2s}{2}} U(y_i) - (\mu_j \mu_i)^{\frac{n-2s}{2}} (H(x, q_i) - \Phi_i^*(x, t)) \right]. \end{aligned} \quad (2.6)$$

Observe that $|\Theta_j| \lesssim \mu_0^{n-2s}$ uniformly in small δ , we assume $U(y_j)^{-1} |\Theta_j| < \frac{1}{2}$ in the considered region. By Taylor expansion, we have

$$S_{21} = \mu_j^{-\frac{n+2s}{2}} \left[p U(y_j)^{p-1} \Theta_j + p(p-1) \int_0^1 (1-s) (U(y_j) + s \Theta_j)^{p-2} ds \Theta_j^2 \right].$$

For $i \neq j$,

$$\begin{aligned} U(y_i) &= U \left(\frac{\mu_j y_j + \xi_j - \xi_i}{\mu_i} \right) = \frac{\alpha_{n,s} \mu_i^{n-2s}}{(\mu_i^2 + |\mu_j y_j + \xi_j - \xi_i|^2)^{\frac{n-2s}{2}}} \\ &= \frac{\alpha_{n,s} \mu_i^{n-2s}}{|\mu_j y_j + \xi_j - \xi_i|^{n-2s}} + \mu_i^{n-2s+2} f(\xi, \mu, \mu_j y_j), \end{aligned}$$

where f is smooth in its arguments and $f(q, 0, 0) = 0$. Then

$$\begin{aligned} \Theta_j &= -\mu_j^{n-2s} (H(\mu_j y_j + \xi_j, q_j) - \Phi_j^*(\mu_j y_j + \xi_j, t)) \\ &\quad + \sum_{i \neq j} \left[(\mu_j \mu_i)^{\frac{n-2s}{2}} G(\mu_j y_j + \xi_j, q_i) + \mu_i^{n-2s+2} f(\xi, \mu, \mu_j y_j) \right. \\ &\quad \left. + (\mu_j \mu_i)^{\frac{n-2s}{2}} \Phi_i^*(\mu_j y_j + \xi_j, t) \right]. \end{aligned}$$

By further expansion, we get

$$\begin{aligned} \Theta_j &= -\mu_j^{n-2s} (H(q_j, q_j) - \Phi_j^*(q_j, t)) + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} G(q_j, q_i) \\ &\quad + \mu_i^{n-2s+2} f(\xi, \mu, \mu_j y_j) + (\mu_j \mu_i)^{\frac{n-2s}{2}} \Phi_i^*(\mu_j y_j + \xi_j, t) + (\mu_j y_j + \xi_j - q_j) \cdot \\ &\quad \left[-\mu_j^{n-2s} \nabla (H(q_j, q_j) - \Phi_j^*(q_j, t)) + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \\ &\quad + \int_0^1 \left\{ -\mu_j^{n-2s} D_x^2 (H - \Phi_j^*) (q_j + s(\mu_j y_j + \xi_j - q_j), q_j) \right. \\ &\quad \left. + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} D_x^2 G(q_j + s(\mu_j y_j + \xi_j - q_j), q_i) \right\} [\mu_j y_j + \xi_j - q_j]^2 (1-s) ds. \end{aligned}$$

We conclude that

$$\begin{aligned} \Theta_j &= -\mu_j^{n-2s} (H(q_j, q_j) - \Phi_j^*(q_j, t)) + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} G(q_j, q_i) \\ &\quad + \left[-\mu_j^{n-2s+1} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}+1} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \cdot y_j \\ &\quad + \mu_0^{n-2s} (\xi_j - q_j) \cdot f(\xi, \mu_j y_j, \mu_0^{-1} \mu) + \mu_0^{n-2s+2} F(\xi, \mu_j y_j, \mu_0^{-1} \mu) [y_j]^2 \\ &\quad + \mu_i^{n-2s+2} f(\xi, \mu, \mu_j y_j), \end{aligned}$$

where f and F are smooth and bounded in its arguments. On the other hand, we have

$$\begin{aligned} S_{22} &= -\sum_{i \neq j} \mu_i^{\frac{n+2s}{2}} U(y_i)^p - \sum_{i=1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t) \\ &= -\sum_{i \neq j} \frac{\alpha_{n,s}^p \mu_i^{\frac{n+2s}{2}}}{|q_j - q_i|^{n+2s}} + \mu_i^{\frac{n+2s}{2}} f(\xi, \mu, \mu_i y_i) - \sum_{i=1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t), \end{aligned}$$

so

$$S_{22} = \mu_0^{\frac{n+2s}{2}} f(\xi, \mu_0^{-1} \mu, \mu_j y_j) - \sum_{i=1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t),$$

where f is smooth in its arguments and $f(q, 0, 0) = 0$.

Decompose $S_1 = S_{11} + S_{12}$, where

$$S_{11} := \mu_j^{-\frac{n-2s}{2}-1} \xi_j \cdot \nabla U(y_j) + \mu_j^{-\frac{n-2s}{2}-1} \mu_j Z_{n+1}(y_j) - \mu_j^{\frac{n-2s}{2}} \partial_t \Phi_j^*(x, t),$$

$$\begin{aligned}
S_{12} &:= \sum_{i \neq j} \mu_i^{-\frac{n-2s}{2}-1} \dot{\xi}_i \cdot \nabla U(y_i) - \left(\partial_t \mu_j^{\frac{n-2s}{2}} \right) \Phi_j^*(x, t) \\
&\quad + \mu_i^{-\frac{n-2s}{2}-1} \dot{\mu}_i Z_{n+1}(y_i) + \sum_{i=1}^k \frac{n-2s}{2} \mu_i^{\frac{n-2s}{2}-1} \dot{\mu}_i H(x, q_i) \\
&\quad - \sum_{i \neq j}^k \partial_t \left(\mu_i^{\frac{n-2s}{2}} \Phi_i^*(x, t) \right).
\end{aligned}$$

We write

$$\begin{aligned}
S_{12} &= \sum_{i \neq j} -\alpha_{n,s}(n-2s) \mu_i^{\frac{n-2s}{2}} \dot{\xi}_i \cdot \left[\frac{q_i - q_j}{|q_i - q_j|^{n-2s+2}} + \vec{f}_i(\xi, \mu, \mu_j y) \right] \\
&\quad + \sum_{i \neq j} \mu_i^{\frac{n-2s}{2}-1} \dot{\mu}_i \left[\frac{c_{n,s}}{|q_i - q_j|^{n-2s}} + f_i(\xi, \mu, \mu_j y) \right] \\
&\quad + \sum_{i=1}^k \frac{n-2s}{2} \mu_i^{\frac{n-2s}{2}-1} \dot{\mu}_i [(H(q_j, q_i) - \Phi_i^*(q_j, t)) + f_i(\mu_j y, \xi)],
\end{aligned}$$

where \vec{f}_i are smooth in their arguments vanishing in the limit. In total, we can write

$$S_{12} = \mu_0^{\frac{n-2s}{2}-1} \sum_{i=1}^k \dot{\mu}_i f_{i0}(\mu_0^{-1} \mu, \xi, \mu_j y) + \mu_0^{\frac{n-2s}{2}} \sum_{i=1}^k \dot{\xi}_i \cdot \vec{f}_{i1}(\mu_0^{-1} \mu, \xi, \mu_j y)$$

for functions f_{i0}, \vec{f}_{i1} smooth in their arguments. This concludes the proof of the lemma. \square

Next, we try to find a solution with the following form

$$u(x, t) = u_{\mu, \xi}(x, t) + \tilde{\phi}(x, t),$$

where $\tilde{\phi}$ is a small term. By $S(u_{\mu, \xi} + \tilde{\phi}) = 0$, our main equation can be expressed with respect to $\tilde{\phi}$ as

$$-\partial_t \tilde{\phi} - (-\Delta)^s \tilde{\phi} + p u_{\mu, \xi}^{p-1} \tilde{\phi} + S(u_{\mu, \xi}) + \tilde{N}_{\mu, \xi}(\tilde{\phi}), \quad (2.7)$$

where

$$\tilde{N}_{\mu, \xi}(\tilde{\phi}) = (u_{\mu, \xi} + \tilde{\phi})^p - u_{\mu, \xi}^p - p u_{\mu, \xi}^{p-1} \tilde{\phi}. \quad (2.8)$$

Write $\tilde{\phi}(x, t)$ in self-similar form around q_j

$$\tilde{\phi}(x, t) = \mu_j^{-\frac{n-2s}{2}} \phi \left(\frac{x - \xi_j}{\mu_j}, t \right). \quad (2.9)$$

By a direct computation, we obtain from (2.7)-(2.9) that

$$0 = \mu_j^{\frac{n+2s}{2}} S(u_{\mu, \xi} + \tilde{\phi}) = -(-\Delta)_y^s \phi + p U(y)^{p-1} \phi + \mu_j^{\frac{n+2s}{2}} S(u_{\mu, \xi}) + A[\phi] \quad (2.10)$$

with

$$\begin{aligned}
A[\phi] &= -\mu_j^{2s} \partial_t \phi + \mu_j^{2s-1} \dot{\mu}_j \left[\frac{n-2s}{2} \phi + y \cdot \nabla_y \phi \right] + \nabla_y \phi \cdot \mu_j^{2s-1} \dot{\xi}_j \\
&\quad + p \left[(U(y) + \Theta_j)^{p-1} - U(y)^{p-1} \right] \phi + (U(y) + \Theta_j + \phi)^p \\
&\quad - (U(y) + \Theta_j)^p - p(U(y) + \Theta_j)^{p-1} \phi,
\end{aligned}$$

where Θ_j is defined in (2.6). We assume that ϕ decays in the y variable and $A[\phi]$ is small when t is large.

2.3. Improvement of the approximation. To improve the approximation, $\phi(y, t)$ should be equal to the solution $\phi_{0j}(y, t)$ of the following elliptic type equation of order $2s$

$$-(-\Delta)_y^s \phi_{0j} + pU(y)^{p-1} \phi_{0j} = -\mu_j^{\frac{n+2s}{2}} S(u_{\mu, \xi}) \text{ in } \mathbb{R}^n, \quad \phi_{0j}(y, t) \rightarrow 0 \text{ as } |y| \rightarrow \infty \quad (2.11)$$

at main order.

Equation (2.11) is a special case of

$$L_0[\psi] := -(-\Delta)_y^s \psi + pU(y)^{p-1} \psi = h(y) \text{ in } \mathbb{R}^n, \quad \psi(y) \rightarrow 0 \text{ as } |y| \rightarrow \infty. \quad (2.12)$$

It is well known (see [19]) that every bounded solution of $L_0[\psi] = 0$ in \mathbb{R}^n is the linear combination of the functions

$$Z_1, \dots, Z_{n+1},$$

where

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \dots, n, \quad Z_{n+1}(y) := \frac{n-2s}{2} U(y) + y \cdot \nabla U(y).$$

Furthermore, problem (2.12) is solvable for $h(y) = O(|y|^{-m})$, $m > 2s$, if it holds that

$$\int_{\mathbb{R}^n} h(y) Z_i(y) dy = 0 \quad \text{for all } i = 1, \dots, n+1.$$

First, we consider the solvability condition for (2.11),

$$\int_{\mathbb{R}^n} \mu_j^{\frac{n+2s}{2}} S(u_{\mu, \xi})(y, t) Z_{n+1}(y) dy = 0. \quad (2.13)$$

We claim that by choosing $\mu_{0j} = b_j \mu_0(t)$ for suitable positive constant b_j , $j = 1, \dots, k$, $\mu_0(t) = c_{n,s} t^{-\frac{1}{n-4s}}$ with $c_{n,s}$ be a positive constant depending only on n and s , this identity can be achieved at main order. Note that, we have $\dot{\mu}_0(t) = -\frac{1}{(n-4s)c_{n,s}^{\frac{n-4s}{2}}} \mu_0^{n-4s+1}(t)$. The main contribution to the integral comes from the term

$$\begin{aligned} E_{0j} = & pU(y_j)^{p-1} \left[-\mu_j^{n-2s-1} H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}-1} \mu_i^{\frac{n-2s}{2}} G(q_j, q_i) \right] \\ & + pU(y_j)^{p-1} \mu_j^{n-2s-1} \Phi_j^0(q_j, t) \\ & + \mu_j^{2s-2} \dot{\mu}_j \left(Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y_j|^2)^{\frac{n-2s}{2}}} \right). \end{aligned}$$

Now, let us compute the nonlocal term $\Phi_j^0(q_j, t)$. Since the heat kernel function $p(t, x)$ satisfies

$$p(t - \tilde{s}, q_j - y) \asymp \frac{t - \tilde{s}}{[(t - \tilde{s})^{\frac{1}{s}} + |q_j - y|^2]^{\frac{n+2s}{2}}},$$

we have

$$p(t - \tilde{s}, q_j - y) = (t - \tilde{s})^{-\frac{n}{2s}} p \left(1, \frac{|q_j - y|}{(t - \tilde{s})^{\frac{1}{2s}}} \right)$$

and

$$\begin{aligned}
\Phi_j^0(q_j, t) &= - \int_{t_0}^t \int_{\mathbb{R}^n} p(t - \tilde{s}, q_j - y) \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \frac{\mu_j^{-(n-2s)}(\tilde{s})}{\left(1 + \left| \frac{y - \xi_j(\tilde{s})}{\mu_j(\tilde{s})} \right|^2\right)^{\frac{n-2s}{2}}} dy d\tilde{s} \\
&= -(1 + o(1)) \int_{t_0}^t \int_{\mathbb{R}^n} p(t - \tilde{s}, q_j - y) \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \frac{\mu_j^{-(n-2s)}(\tilde{s})}{\left(1 + \left| \frac{y - q_j}{\mu_j(\tilde{s})} \right|^2\right)^{\frac{n-2s}{2}}} dy d\tilde{s} \\
&= -(1 + o(1)) \int_{t_0}^t \frac{1}{(t - \tilde{s})^{\frac{n}{2s}}} d\tilde{s} \int_{\mathbb{R}^n} p\left(1, \frac{q_j - y}{(t - \tilde{s})^{\frac{1}{2s}}}\right) \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \\
&\quad \times \frac{\mu_j^{-(n-2s)}(\tilde{s}) (t - \tilde{s})^{\frac{n}{2s}}}{\left(1 + \left| \frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})} \frac{q_j - y}{(t - \tilde{s})^{\frac{1}{2s}}} \right|^2\right)^{\frac{n-2s}{2}}} d \frac{y - q_j}{(t - \tilde{s})^{\frac{1}{2s}}} \\
&= -(1 + o(1)) \int_{t_0}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) d\tilde{s} \int_{\mathbb{R}^n} p\left(1, \frac{q_j - y}{(t - \tilde{s})^{\frac{1}{2s}}}\right) \\
&\quad \times \frac{1}{\left(1 + \left| \frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})} \frac{q_j - y}{(t - \tilde{s})^{\frac{1}{2s}}} \right|^2\right)^{\frac{n-2s}{2}}} d \frac{y - q_j}{(t - \tilde{s})^{\frac{1}{2s}}} \\
&= -(1 + o(1)) \int_{t_0}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s},
\end{aligned}$$

where

$$F(a) = \int_{\mathbb{R}^n} p(1, x) \frac{1}{(1 + a^2|x|^2)^{\frac{n-2s}{2}}} dx.$$

We claim that

$$\Phi_j^0(q_j, t) = -(1 + o(1)) \int_{t_0}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} = c(1 + o(1)) \quad (2.14)$$

for a suitable positive constant c depending on n , s and b_j , $j = 1, \dots, k$.

Indeed, for a small constant $\delta > 0$, we decompose the integral

$$\int_{t_0}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s}$$

as

$$\begin{aligned}
&\int_{t_0}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} \\
&= \int_{t_0}^{t-\delta} \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} \\
&\quad + \int_{t-\delta}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t - \tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} \\
&:= I_1 + I_2.
\end{aligned}$$

For the first integral we have $t - \tilde{s} > \delta$, therefore

$$\begin{aligned} 0 \leq -I_1 &\leq \int_{t_0}^{t-\delta} \mu_j^{-2s}(\tilde{s}) F\left(\frac{(t-\tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} \leq C \int_{t_0}^{t-\delta} \mu_j^{-2s}(\tilde{s}) \left|\frac{(t-\tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right|^{-(n-2s)} d\tilde{s} \\ &= C b_j^{n-4s} c_{n,s}^{n-4s} \int_{t_0}^{t-\delta} \frac{1}{\tilde{s}} \frac{1}{(t-\tilde{s})^{\frac{n-2s}{2s}}} d\tilde{s} \leq \frac{C b_j^{n-4s} c_{n,s}^{n-4s}}{t_0} \frac{2s}{n-4s} \frac{1}{\delta^{\frac{n-4s}{2s}}}. \end{aligned}$$

Here we have used the ansatz $\mu_{0j} = b_j c_{n,s} t^{-\frac{1}{n-4s}}$ and the fact that $|a|^{n-2s} F(a) \leq C$. For the second integral

$$I_2 = \int_{t-\delta}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t-\tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s},$$

using change of variables $\frac{(t-\tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})} = \hat{s}$, we obtain

$$d\tilde{s} = -\frac{\mu_j(\tilde{s})}{\frac{1}{2s}(t-\tilde{s})^{\frac{1}{2s}-1} + \dot{\mu}_j(\tilde{s})\hat{s}} d\hat{s}$$

and the integral becomes

$$\begin{aligned} I_2 &= \int_{t-\delta}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t-\tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} \\ &= \int_0^{\frac{\delta^{\frac{1}{2s}}}{\mu_j(t-\delta)}} \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F(\hat{s}) \frac{\mu_j(\tilde{s})}{\frac{1}{2s}(t-\tilde{s})^{\frac{1}{2s}-1} + \dot{\mu}_j(\tilde{s})\hat{s}} d\hat{s}. \end{aligned}$$

Note that $\frac{1}{2s}(t-\tilde{s})^{\frac{1}{2s}-1} + \dot{\mu}_j(\tilde{s})\hat{s} = \frac{1}{2s}(t-\tilde{s})^{\frac{1}{2s}-1} \left(1 - \frac{2s}{(n-4s)\tilde{s}}(t-\tilde{s})\right) > \frac{1}{2s}(t-\tilde{s})^{\frac{1}{2s}-1} \left(1 - \frac{2s}{(n-4s)\tilde{s}}\delta\right)$, $d\tilde{s} = \frac{\mu_j(\tilde{s})}{\frac{1}{2s}(t-\tilde{s})^{\frac{1}{2s}-1}} (1 + O(\delta)) d\hat{s}$ for δ small and

$$I_2 = -\frac{2s b_j^{4s-n}}{(n-4s) c_{n,s}^{n-4s}} \left(\int_0^{\frac{\delta^{\frac{1}{2s}}}{\mu_j(t-\delta)}} \hat{s}^{2s-1} F(\hat{s}) d\hat{s} + o(1) \right) = -\frac{2s b_j^{4s-n}}{(n-4s) c_{n,s}^{n-4s}} A + o(1)$$

as long as $\frac{\delta^{\frac{1}{2s}}}{\mu_j(t-\delta)}$ is large. Here $A = \int_0^\infty \hat{s}^{2s-1} F(\hat{s}) d\hat{s} < +\infty$ since $n > 4s$. Thus we have

$$\begin{aligned} \Phi_j^0(q_j, t) &= -(1 + o(1)) \int_{t_0}^t \frac{\dot{\mu}_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-(n-2s)}(\tilde{s}) F\left(\frac{(t-\tilde{s})^{\frac{1}{2s}}}{\mu_j(\tilde{s})}\right) d\tilde{s} \\ &= \frac{2s b_j^{4s-n}}{(n-4s) c_{n,s}^{n-4s}} A + o(1) := B b_j^{4s-n} + o(1) \end{aligned} \tag{2.15}$$

when t_0 is chosen sufficiently large. Here $B = B_{n,s} := \frac{2s}{(n-4s) c_{n,s}^{n-4s}} A$. This proves (2.14).

By direct computations, we have

$$\begin{aligned} & \mu_0^{-(n-2s-1)}(t) \int_{\mathbb{R}^n} E_{0j}(y, t) Z_{n+1}(y) dy \\ & \approx c_1 \left[b_j^{n-2s-1} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}-1} b_i^{\frac{n-2s}{2}} G(q_j, q_i) \right] \\ & \quad - \frac{2sc_1 A + c_2}{(n-4s)c_{n,s}^{n-4s}} b_j^{2s-1} \end{aligned} \quad (2.16)$$

with

$$\begin{aligned} c_1 &= -p \int_{\mathbb{R}^n} U(y)^{p-1} Z_{n+1}(y) dy, \\ c_2 &= \int_{\mathbb{R}^n} \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) Z_{n+1}(y) dy. \end{aligned}$$

Denote

$$\mu_j(t) = b_j \mu_0(t) = b_j c_{n,s} t^{-\frac{1}{n-4s}}.$$

Then the solvability conditions (2.13) can be achieved at main order if

$$b_j^{n-2s} H(q_j, q_j) - \sum_{i \neq j} (b_i b_j)^{\frac{n-2s}{2}} G(q_j, q_i) - \frac{2sc_1 A + c_2}{(n-4s)c_{n,s}^{n-4s} c_1} b_j^{2s} = 0 \text{ for all } j = 1, \dots, k. \quad (2.17)$$

By imposing $-\frac{2sc_1 A + c_2}{(n-4s)c_{n,s}^{n-4s} c_1} = -\frac{2s}{n-2s}$, namely

$$c_{n,s} = \left[\frac{(2sc_1 A + c_2)(n-2s)}{2s(n-4s)c_1} \right]^{\frac{1}{n-4s}},$$

we have

$$\dot{\mu}_0(t) = -\frac{2sc_1}{(2sc_1 A + c_2)(n-2s)} \mu_0^{n-4s+1}(t). \quad (2.18)$$

From (2.17) and (2.18), the constants b_j satisfy the system

$$b_j^{n-2s-1} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}-1} b_i^{\frac{n-2s}{2}} G(q_j, q_i) = \frac{2sb_j^{2s-1}}{n-2s} \text{ for all } j = 1, \dots, k. \quad (2.19)$$

System (2.19) is equivalent to the variational problem $\nabla_b I(b) = 0$ with

$$I(b) := \frac{1}{n-2s} \left[\sum_{j=1}^k b_j^{n-2s} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) - \sum_{j=1}^k b_j^{2s} \right]. \quad (2.20)$$

Let $\Lambda_j = b_j^{\frac{n-2s}{2}}$. Then

$$(n-2s)I(b) = \tilde{I}(\Lambda) = \left[\sum_{j=1}^k H(q_j, q_j) \Lambda_j^2 - \sum_{i \neq j} G(q_j, q_i) \Lambda_i \Lambda_j - \sum_{j=1}^k \Lambda_j^{\frac{4s}{n-2s}} \right].$$

Standard argument shows that system (2.19) has a unique solution if and only if the following matrix

$$\mathcal{G}(q) = \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_2, q_1) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -G(q_k, q_1) & -G(q_k, q_2) & \cdots & H(q_k, q_k) \end{bmatrix}$$

is positive definite.

Next, we consider the solvability conditions for (2.11),

$$\int_{\mathbb{R}^n} \mu_j^{\frac{n+2s}{2}} S(u_{\mu, \xi})(y, t) Z_i(y) dy = 0, \quad i = 1, \dots, n.$$

It is clear that these conditions are fulfilled at main order by simply setting $\xi_{0j} = q_j$.

Now we fix the function $\mu_0(t)$ and the positive constants b_j satisfying (2.19) and denote

$$\bar{\mu}_0 = (\mu_{01}, \dots, \mu_{0k}) = (b_1 \mu_0, \dots, b_k \mu_0).$$

Let Φ_j be the unique solution to (2.11) for $\mu = \bar{\mu}_0$. Then we have

$$-(-\Delta)_y^s \Phi_j + pU(y)^{p-1} \Phi_j = -\mu_{0j} E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}] \text{ in } \mathbb{R}^n, \quad \Phi_j(y, t) \rightarrow 0 \text{ as } |y| \rightarrow \infty.$$

From the definition of μ_0 and b_j , one has

$$\mu_{0j} E_{0j} = -\gamma_j \mu_0^{n-2s} q_0(y),$$

where the constant γ_j is positive and

$$q_0(y) := \frac{pU(y)^{p-1} c_2 b_j^{2s}}{(n-4s) c_{n,s}^{n-4s} c_1} + \frac{b_j^{2s}}{(n-4s) c_{n,s}^{n-4s}} \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \quad (2.21)$$

with $\int_{\mathbb{R}^n} q_0(y) Z_{n+1} dy = 0$.

Let $p_0 = p_0(|y|)$ be the radial solution of $L_0(p_0) = q_0$. Then $p_0(|y|) = O(|y|^{-2s})$ as $|y| \rightarrow \infty$ since we have (2.21). Therefore,

$$\Phi_j(y, t) = \gamma_j \mu_0^{n-2s} p_0(y). \quad (2.22)$$

Thus we can define the corrected approximation as

$$u_{\mu, \xi}^*(x, t) = u_{\mu, \xi}(x, t) + \tilde{\Phi}(x, t) \quad (2.23)$$

with

$$\tilde{\Phi}(x, t) = \sum_{j=1}^k \mu_j^{-\frac{n-2s}{2}} \Phi_j \left(\frac{x - \xi_j}{\mu_j}, t \right).$$

2.4. Estimating the error $S(u_{\mu, \xi}^*)$. In the region $|x - q_i| > \delta$ for all $i = 1, \dots, k$, $S(u_{\mu, \xi}^*)$ can be described as

$$S(u_{\mu, \xi}^*)(x, t) = \mu_0^{\frac{n-2s}{2}-1} \sum_{j=1}^k \dot{\mu}_j f_j + \mu_0^{\frac{n-2s}{2}} \sum_{j=1}^k \dot{\xi}_j \cdot \vec{f}_j + \mu_0^{\frac{n+2s}{2}} f, \quad (2.24)$$

where the functions f_j, \vec{f}_j, f are smooth bounded and depend on $(x, \mu_0^{-1} \mu, \xi)$. Now we consider the region near each of the points q_j . By direct computations, $S(u_{\mu, \xi}^*)$

is given by

$$\begin{aligned}
S(u_{\mu,\xi}^*) &= S(u_{\mu,\xi}) - \sum_{j=1}^k \mu_j^{-\frac{n+2s}{2}} \mu_{0j} E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}] + \sum_{j=1}^k \mu_j^{-\frac{n+2s}{2}} \left\{ -\mu_j^{2s} \partial_t \Phi_j(y_j, t) \right. \\
&\quad \left. + \mu_j^{2s-1} \dot{\mu}_j \left[\frac{n-2s}{2} \Phi_j(y_j, t) + y_j \cdot \nabla_y \Phi_j \right] + \mu_j^{2s-1} \nabla_y \Phi_j(y_j, t) \cdot \dot{\xi}_j \right\} \\
&\quad + (u_{\mu,\xi} + \tilde{\Phi})^p - u_{\mu,\xi}^p - p \sum_{j=1}^k \mu_j^{-\frac{n+2s}{2}} U(y_j)^{p-1} \Phi_j(y_j, t),
\end{aligned} \tag{2.25}$$

where $y_j = \frac{x - \xi_j}{\mu_j}$.

From (2.10), for a given fixed j and $|x - q_j| \leq \delta$, we have

$$\mu_j^{\frac{n+2s}{2}} S(u_{\mu,\xi}^*) = \mu_j^{\frac{n+2s}{2}} S(u_{\mu,\xi}) - \mu_{0j} E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}] + A_j(y). \tag{2.26}$$

After direct computations,

$$A_j = \mu_0^{n+4s} f(\mu_0^{-1} \mu, \xi, \mu_j y) + \frac{\mu_0^{2n-4s}}{1 + |y_j|^{2s}} g(\mu_0^{-1} \mu, \xi, \mu_j y), \quad y_j = \frac{x - \xi_j}{\mu_j}, \tag{2.27}$$

where f and g are smooth and bounded.

Finally we set

$$\mu(t) = \bar{\mu}_0 + \lambda(t) \text{ with } \lambda(t) = (\lambda_1(t), \dots, \lambda_k(t)).$$

From (2.26), we have

$$\begin{aligned}
S(u_{\mu,\xi}^*) &= \mu_j^{-\frac{n+2s}{2}} \left\{ \mu_{0j} (E_{0j}[\mu, \dot{\mu}_j] - E_{0j}[\bar{\mu}_0, \dot{\mu}_{0j}]) + \lambda_j E_{0j}[\mu, \dot{\mu}_j] \right. \\
&\quad \left. + \mu_j E_{1j}[\mu, \dot{\xi}_j] + R_j + A_j \right\}.
\end{aligned}$$

Recall Lemma 2.1 that

$$\begin{aligned}
E_{0j}[\mu, \dot{\mu}_j] &= pU(y)^{p-1} \left[-\mu_j^{n-2s-1} H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}-1} \mu_i^{\frac{n-2s}{2}} G(q_j, q_i) \right] \\
&\quad + pU(y)^{p-1} \mu_j^{n-2s-1} \Phi_j^0(q_j, t) + \mu_j^{2s-2} \dot{\mu}_j \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y|^2)^{\frac{n-2s}{2}}} \right).
\end{aligned}$$

Note that Φ_j^0 depends on μ , $\dot{\mu}$ and

$$\begin{aligned}
\mu_0^{n-2s-1} \Phi_j^0[\bar{\mu}_0 + \lambda, b_j \dot{\mu}_0 + \dot{\lambda}_j](q_j, t) &- \mu_0^{n-2s-1} \Phi_j^0[\bar{\mu}_0, b_j \dot{\mu}_0](q_j, t) \\
&= -(b_j \mu_0)^{2s-2} 2s A \dot{\lambda}_j - \mu_0^{n-2s-2} (n-4s+1) b_j^{2s-2} B \lambda_j
\end{aligned}$$

which can be deduced by similar arguments as (2.15), we have

$$\begin{aligned}
 & E_{0j}[\bar{\mu}_0 + \lambda, b_j \dot{\mu}_0 + \dot{\lambda}_j] - E_{0j}[\bar{\mu}_0, b_j \dot{\mu}_0] \\
 &= (b_j \mu_0)^{2s-2} \dot{\lambda}_j \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \\
 &+ (2s-2)(b_j \mu_0)^{2s-3} \lambda_j (b_j \dot{\mu}_0 + \dot{\lambda}_j) \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \\
 &- \mu_0^{n-2s-2} pU(y)^{p-1} \left[\sum_{i=1}^k M_{ij} \lambda_i + \sum_{i,l=1}^k f_{ijl}(\mu_0^{-1} \lambda) \lambda_i \lambda_l \right] \\
 &+ \mu_0^{n-2s-2} pU(y)^{p-1} (n-2s-1) b_j^{2s-2} B \lambda_j \\
 &- pU(y)^{p-1} (b_j \mu_0)^{2s-2} 2s A \dot{\lambda}_j - \mu_0^{n-2s-2} pU(y)^{p-1} (n-4s+1) b_j^{2s-2} B \lambda_j,
 \end{aligned}$$

where f_{ijl} are smooth functions and for $i = j$,

$$M_{ij} = (n-2s-1) b_j^{n-2s-2} H(q_j, q_j) - \left(\frac{n-2s}{2} - 1 \right) \sum_{i \neq j} b_j^{\frac{n-2s}{2}-2} b_i^{\frac{n-2s}{2}} G(q_j, q_i),$$

for $i \neq j$,

$$M_{ij} = -\frac{n-2s}{2} \sum_{i \neq j} b_j^{\frac{n-2s}{2}-1} b_i^{\frac{n-2s}{2}-1} G(q_j, q_i).$$

$M = D^2 I_0(b)$ with

$$I_0(b) := \frac{1}{n-2s} \left[\sum_{j=1}^k b_j^{n-2s} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) \right].$$

Since $D^2 I(b)$ is positive definite, we denote its positive eigenvalues corresponding to the eigenvectors \bar{w}_j by $\bar{\sigma}_j$ for $j = 1, \dots, k$. Thus

$$D^2 I(b) = P^T \text{diag}(\bar{\sigma}_1, \dots, \bar{\sigma}_k) P \quad (2.28)$$

and P is the $k \times k$ matrix given by $P := (\bar{w}_1, \dots, \bar{w}_k)$. From the definition of b_j in (2.19), one has

$$\begin{aligned}
 M = D^2 I_0(b) &= D^2 I(b) + \frac{2s(2s-1)}{n-2s} \text{diag}(b_1^{2s-2}, \dots, b_k^{2s-2}) \\
 &= P^T \text{diag}(\bar{\sigma}_1 + \frac{2s(2s-1)}{n-2s} b_1^{2s-2}, \dots, \bar{\sigma}_k + \frac{2s(2s-1)}{n-2s} b_k^{2s-2}) P.
 \end{aligned}$$

Now we estimate $\lambda_j E_{0j}[\mu, \dot{\mu}_j]$. Indeed, we have

$$\begin{aligned}
& \lambda_j E_{0j}[\mu, \dot{\mu}_j] \\
&= (b_j \mu_0)^{2s-2} \lambda_j \dot{\lambda}_j \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \\
&\quad + (2s-2)(b_j \mu_0)^{2s-3} \lambda_j^2 (b_j \dot{\mu}_0 + \dot{\lambda}_j) \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \\
&\quad + \lambda_j b_j^{2s-1} \left[\mu_0^{2s-2} \dot{\mu}_0 \left(Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n-2s}{2}}} \right) \right. \\
&\quad \left. + pU(y)^{p-1} \mu_0^{n-2s-1} \left(-b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j^{\frac{n-6s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) \right) \right] \\
&\quad + pU(y)^{p-1} b_j^{2s-1} \mu_0^{n-2s-1} B \lambda_j \\
&\quad - \mu_0^{n-2s-2} pU(y)^{p-1} \sum_{i,l=1}^k f_{ijl}(\mu_0^{-1} \lambda) \lambda_i \lambda_l,
\end{aligned}$$

where functions f_{ijl} are smooth in its arguments.

Collecting all the above estimates, we have the full expansion for $S(u_{\mu, \xi}^*)$.

Lemma 2.2. *We consider the region $|x - q_j| \leq \frac{1}{2} \min_{i \neq l} |q_i - q_l|$ for a fixed index j . Let $\mu = \bar{\mu}_0 + \lambda$ and $|\lambda(t)| \leq \mu_0(t)^{1+\sigma}$ for some $0 < \sigma < \bar{\sigma}$ with $\bar{\sigma}$ be a constant satisfying $0 < \bar{\sigma} \leq \frac{n-2s}{2s} \bar{\sigma}_j b_j^{2-2s}$, $j = 1, \dots, k$. Then for large t , $S(u_{\mu, \xi}^*)$ can be expressed as*

$$\begin{aligned}
S(u_{\mu, \xi}^*) &= \sum_{j=1}^k \mu_j^{-\frac{n+2s}{2}} \\
&\quad \left\{ \mu_{0j} (b_j \mu_0)^{2s-2} \dot{\lambda}_j \left(Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y_j|^2)^{\frac{n-2s}{2}}} - 2s A p U(y_j)^{p-1} \right) \right. \\
&\quad - \mu_{0j} \mu_0^{n-2s-2} pU(y_j)^{p-1} \sum_{i=1}^k M_{ij} \lambda_i + \mu_j^{2s-1} \alpha_{n,s} (n-2s) \frac{\dot{\xi}_j \cdot y_j}{(1+|y_j|^2)^{\frac{n-2s}{2}+1}} \\
&\quad \left. + \mu_j pU(y_j)^{p-1} \left[-\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \cdot y_j \right\} \\
&\quad + \sum_{j=1}^k \mu_j^{-\frac{n+2s}{2}} \lambda_j b_j^{2s-1} \\
&\quad \left[(2s-1) \mu_0^{2s-2} \dot{\mu}_0 \left(Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y_j|^2)^{\frac{n-2s}{2}}} \right) \right. \\
&\quad \left. + pU(y_j)^{p-1} \mu_0^{n-2s-1} \left(-b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j^{\frac{n-6s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) \right) \right]
\end{aligned}$$

$$\begin{aligned}
 & \left. + pU(y_j)^{p-1} \mu_0^{n-2s-1} (2s-1)B \right] \\
 & + \mu_0^{-\frac{n+2s}{2}} \left[\sum_{j=1}^k \frac{\mu_0^{n-2s+2} g_j}{1+|y_j|^{4s-2}} + \sum_{j=1}^k \frac{\mu_0^{2n-4s} g_j}{1+|y_j|^{2s}} + \sum_{j=1}^k \frac{\mu_0^{n-2s} g_j}{1+|y_j|^{4s}} \lambda_j \right] \\
 & + \mu_0^{-\frac{n+2s}{2}} \left[\sum_{j=1}^k \frac{\mu_0^{n-2s} \vec{g}_j}{1+|y_j|^{4s}} \cdot (\xi_j - q_j) \right] \\
 & + \mu_0^{-\frac{n+2s}{2}} \left[\mu_0^{n-2s-2} \sum_{i,j,l=1}^k pU(y_j)^{p-1} f_{ijl} \lambda_i \lambda_l + \sum_{i,j,l=1}^k \frac{f_{ijl}}{1+|y_j|^{n-2s}} \lambda_i \dot{\lambda}_l \right] \\
 & + \mu_0^{-\frac{n+2s}{2}} \left[\mu_0^{n+2s} f + \mu_0^{n-1} \sum_{i=1}^k \dot{\mu}_i f_i + \mu_0^n \sum_{i=1}^k \dot{\xi}_i \vec{f}_i \right],
 \end{aligned}$$

where $x = \xi_j + \mu_j y_j$, \vec{f}_i , f_i , f , f_{ijl} are smooth and bounded functions depending on $(\mu_0^{-1} \mu, \xi, x)$ and g_j , \vec{g}_j depend on $(\mu_0^{-1} \mu, \xi, y_j)$.

3. THE INNER-OUTER GLUING PROCEDURE

We are looking for a solution to (2.1) with the form

$$u = u_{\mu, \xi}^* + \tilde{\phi}$$

when t_0 is sufficiently large. The function $\tilde{\phi}(x, t)$ is a smaller term and we will find it by means of the **inner-outer gluing procedure**.

Let us write

$$\tilde{\phi}(x, t) = \psi(x, t) + \phi^{in}(x, t) \quad \text{where} \quad \phi^{in}(x, t) := \sum_{j=1}^k \eta_{j,R}(x, t) \tilde{\phi}_j(x, t)$$

with

$$\tilde{\phi}_j(x, t) := \mu_{0j}^{-\frac{n-2s}{2}} \phi_j \left(\frac{x - \xi_j}{\mu_{0j}}, t \right), \quad \mu_{0j}(t) = b_j \mu_0(t)$$

and

$$\eta_{j,R}(x, t) = \eta \left(\frac{|x - \xi_j|}{R \mu_{0j}} \right).$$

Here $\eta(\tau)$ is a smooth cut-off function defined on $[0, +\infty)$ with $\eta(\tau) = 1$ for $0 \leq \tau < 1$ and $\eta(\tau) = 0$ for $\tau > 2$. The number R is defined as

$$R = t_0^\rho \quad \text{with} \quad 0 < \rho \ll 1. \quad (3.1)$$

Problem (2.1) can be written with respect to $\tilde{\phi}$ as

$$\begin{cases} \partial_t \tilde{\phi} = -(-\Delta)^s \tilde{\phi} + p(u_{\mu, \xi}^*)^{p-1} \tilde{\phi} + \tilde{N}(\tilde{\phi}) + S(u_{\mu, \xi}^*) & \text{in } \Omega \times (t_0, \infty), \\ \tilde{\phi} = -u_{\mu, \xi}^* & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \end{cases} \quad (3.2)$$

where $\tilde{N}(\tilde{\phi}) = (u_{\mu, \xi}^* + \tilde{\phi})^p - p(u_{\mu, \xi}^*)^{p-1} \tilde{\phi} - (u_{\mu, \xi}^*)^p$ and $S(u_{\mu, \xi}^*) = -\partial_t u_{\mu, \xi}^* - (-\Delta)^s u_{\mu, \xi}^* + (u_{\mu, \xi}^*)^p$.

According to Lemma 2.2, we let $y_j = \frac{x-\xi_j}{\mu_j}$ and denote

$$S(u_{\mu,\xi}^*) = \sum_{j=1}^k S_{\mu,\xi,j} + S_{\mu,\xi}^{(2)}$$

where

$$\begin{aligned} S_{\mu,\xi,j} = & \mu_j^{-\frac{n+2s}{2}} \left\{ \mu_{0j} (b_j \mu_0)^{2s-2} \dot{\lambda}_j \right. \\ & \times \left(Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y_j|^2)^{\frac{n-2s}{2}}} - 2sApU(y_j)^{p-1} \right) \\ & - \mu_{0j} \mu_0^{n-2s-2} pU(y_j)^{p-1} \sum_{i=1}^k M_{ij} \lambda_i \\ & + \lambda_j b_j^{2s-1} \left[(2s-1) \mu_0^{2s-2} \dot{\mu}_0 \left(Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y_j|^2)^{\frac{n-2s}{2}}} \right) \right. \\ & + pU(y_j)^{p-1} \mu_0^{n-2s-1} \left(-b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j^{\frac{n-6s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) \right) \\ & \left. + pU(y_j)^{p-1} \mu_0^{n-2s-1} (2s-1) B \right] \\ & + \mu_j \left[\mu_j^{2s-2} \alpha_{n,s} (n-2s) \frac{\dot{\xi}_j \cdot y_j}{(1+|y_j|^2)^{\frac{n-2s}{2}+1}} \right. \\ & \left. + pU(y_j)^{p-1} \left(-\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right) \cdot y_j \right] \left. \right\}. \end{aligned}$$

Set

$$V_{\mu,\xi} = p \sum_{j=1}^k \left((u_{\mu,\xi}^*)^{p-1} - \left[\mu_j^{-\frac{n-2s}{2}} U \left(\frac{x-\xi_j}{\mu_j} \right) \right]^{p-1} \right) \eta_{j,R} + p \left(1 - \sum_{j=1}^k \eta_{j,R} \right) (u_{\mu,\xi}^*)^{p-1}. \quad (3.3)$$

Then $\tilde{\phi}$ solves problem (3.2) if

(1) ψ solves the **outer problem**

$$\left\{ \begin{aligned} \partial_t \psi &= -(-\Delta)^s \psi + V_{\mu,\xi} \psi \\ &+ \sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \\ &+ \tilde{N}_{\mu,\xi}(\tilde{\phi}) + S_{out}, \quad \text{in } \Omega \times (t_0, \infty), \\ \psi &= -u_{\mu,\xi}^* \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \end{aligned} \right. \quad (3.4)$$

where

$$S_{out} = S_{\mu,\xi}^{(2)} + \sum_{j=1}^k (1 - \eta_{j,R}) S_{\mu,\xi,j} \quad (3.5)$$

and

$$[-(-\Delta)^{\frac{s}{2}}\eta_{j,R}, -(-\Delta)^{\frac{s}{2}}\tilde{\phi}_j] := C_{n,s} \int_{\mathbb{R}^n} \frac{[\eta_{j,R}(y) - \eta_{j,R}(x)][\tilde{\phi}_j(x) - \tilde{\phi}_j(y)]}{|x-y|^{n+2s}} dy. \quad (3.6)$$

(2) $\tilde{\phi}_j$ solves

$$\eta_{j,R} \partial_t \tilde{\phi}_j = \eta_{j,R} \left[-(-\Delta)^s \tilde{\phi}_j + p U_j^{p-1} \tilde{\phi}_j + p U_j^{p-1} \psi + S_{\mu,\xi,j} \right] \text{ in } \mathbb{R}^n \times (t_0, \infty), \quad (3.7)$$

for all $j = 1, \dots, k$, $U_j := \mu_j^{-\frac{n-2s}{2}} U\left(\frac{x-\xi_j}{\mu_j}\right)$. In terms of $\phi_j(y, t)$, (3.7) becomes the **inner problem**

$$\left\{ \begin{array}{l} \mu_{0j}^{2s} \partial_t \phi_j = -(-\Delta)_y^s \phi_j + p U^{p-1}(y) \phi_j \\ \quad + \left\{ \mu_{0j}^{\frac{n+2s}{2}} S_{\mu,\xi,j}(\xi_j + \mu_{0j} y, t) + p \mu_{0j}^{\frac{n-2s}{2}} \frac{\mu_{0j}^{2s}}{\mu_j^{2s}} U^{p-1}\left(\frac{\mu_{0j}}{\mu_j} y\right) \psi(\xi_j + \mu_{0j} y, t) \right. \\ \quad \left. + B_j[\phi_j] + B_j^0[\phi_j] \right\} \chi_{B_{2R}(0)}(y) \text{ in } \mathbb{R}^n \times (t_0, \infty), \end{array} \right. \quad (3.8)$$

for $j = 1, \dots, k$, where

$$B_j[\phi_j] := \mu_{0j}^{2s-1} \mu_{0j} \left(\frac{n-2s}{2} \phi_j + y \cdot \nabla_y \phi_j \right) + \mu_{0j}^{2s-1} \nabla \phi_j \cdot \dot{\xi}_j \quad (3.9)$$

and

$$B_j^0[\phi_j] := p \left[U^{p-1}\left(\frac{\mu_{0j}}{\mu_j} y\right) - U^{p-1}(y) \right] \phi_j + p \left[\mu_{0j}^{2s} (u_{\mu,\xi}^*)^{p-1} - U^{p-1}\left(\frac{\mu_{0j}}{\mu_j} y\right) \right] \phi_j. \quad (3.10)$$

Here $\chi_{B_{2R}(0)}(y)$ is the characteristic function of $B_{2R}(0)$.

The rest of the paper is organized as follows. In Section 4, we solve the outer problem (3.4). In Section 5.1, a linear theory for the inner problem (3.8) is developed. We study the solvability conditions for (3.8) in Section 5.2 and the full problem is finally solved in Section 6.

4. THE OUTER PROBLEM

In this section, we shall solve the outer problem for a given smooth function ϕ which is sufficiently small. We consider the initial boundary value problem

$$\left\{ \begin{array}{l} \partial_t \psi = -(-\Delta)^s \psi + V_{\mu,\xi} \psi \\ \quad + \sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \\ \quad + \tilde{N}_{\mu,\xi}(\tilde{\phi}) + S_{out}, \quad \text{in } \Omega \times (t_0, \infty), \\ \psi = -u_{\mu,\xi}^* \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\ \psi(\cdot, t_0) = \psi_0 \quad \text{in } \mathbb{R}^n \end{array} \right. \quad (4.1)$$

with a smooth and sufficiently small initial condition ψ_0 .

4.1. The model problem. To solve problem (4.1), we first consider the linear problem

$$\begin{cases} \partial_t \psi = -(-\Delta)^s \psi + V_{\mu, \xi} \psi + f(x, t) & \text{in } \Omega \times (t_0, \infty), \\ \psi = g & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\ \psi(\cdot, t_0) = h & \text{in } \mathbb{R}^n, \end{cases} \quad (4.2)$$

where $f(x, t)$, $g(x, t)$ and $h(x)$ are given smooth functions and $V_{\mu, \xi}$ is defined in (3.3). Furthermore, we assume f satisfies

$$|f(x, t)| \leq M \sum_{j=1}^k \frac{\mu_j^{-2s} t^{-\beta}}{1 + |y_j|^{2s+\alpha}}, \quad y_j = \frac{|x - \xi_j|}{\mu_j} \quad (4.3)$$

for $\alpha, \beta > 0$. Denote the least $M > 0$ in (4.3) by $\|f\|_{*, \beta, 2s+\alpha}$.

In what follows, we use the symbol $a \lesssim b$ to denote $a \leq Cb$ for a positive constant C independent of t and t_0 . Then we have the following *a priori* estimate for the model problem (4.2).

Lemma 4.1. *Suppose that $\|f\|_{*, \beta, 2s+\alpha} < +\infty$ for some $\alpha, \beta > 0$, $0 < \alpha \ll 1$, $\|h\|_{L^\infty(\mathbb{R}^n)} < +\infty$ and $\|\tau^\beta g(x, \tau)\|_{L^\infty((\mathbb{R}^n \setminus \Omega) \times (t_0, \infty))} < +\infty$. Let $\phi = \psi[f, g, h]$ be the solution of problem (4.2). Then there exists $\delta = \delta(\Omega) > 0$ small such that for all (x, t) we have*

$$\begin{aligned} |\psi(x, t)| &\lesssim \|f\|_{*, \beta, 2s+\alpha} \left(\sum_{j=1}^k \frac{t^{-\beta}}{1 + |y_j|^\alpha} \right) \\ &\quad + e^{-\delta(t-t_0)} \|h\|_{L^\infty(\mathbb{R}^n)} + t^{-\beta} \|\tau^\beta g(x, \tau)\|_{L^\infty((\mathbb{R}^n \setminus \Omega) \times (t_0, \infty))}, \end{aligned} \quad (4.4)$$

where $y_j = \frac{x - \xi_j}{\mu_j}$. Moreover, the following Hölder estimate

$$[\psi(\cdot, t)]_{\eta, B_{\mu_j}(\xi_j)} \lesssim \|f\|_{*, \beta, 2s+\alpha} \left(\sum_{j=1}^k \frac{\mu_j^{-\eta} t^{-\beta}}{1 + |y_j|^{\alpha+\eta}} \right) \quad (4.5)$$

holds for some $\eta \in (0, 1)$ and $|y_j| \leq 2R$. Here

$$[\psi(\cdot, t)]_{\eta, B_{\mu_j}(\xi_j)} := \sup_{x, y \in B_{\mu_j}(\xi_j)} \frac{|\psi(x, t) - \psi(y, t)|}{|x - y|^\eta}$$

is the Hölder seminorm.

Proof. Let $\psi_0[g, h]$ be the solution of the fractional heat equation

$$\begin{cases} \partial_t \psi_0 = -(-\Delta)^s \psi_0 & \text{in } \Omega \times (t_0, \infty), \\ \psi_0 = g & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\ \psi_0(\cdot, t_0) = h & \text{in } \mathbb{R}^n. \end{cases} \quad (4.6)$$

Let $v(x)$ be the bounded solution of $-(-\Delta)^s v + 1 = 0$ in Ω with $v = 1$ on $\mathbb{R}^n \setminus \Omega$. Then $v \geq 1$ in Ω and by direct computations, the function

$$\bar{\psi}(x, t) = \left(e^{-\delta(t-t_0)} \|h\|_{L^\infty(\mathbb{R}^n)} + t^{-\beta} \|\tau^\beta g(x, \tau)\|_{L^\infty((\mathbb{R}^n \setminus \Omega) \times (t_0, \infty))} \right) v(x)$$

is a supersolution to (4.6) if $\delta = \delta(\Omega) > 0$ is sufficiently small. Then $|\psi_0| \leq \bar{\psi}$ by the maximum principle (see, for example, [5] and [6]). Thus, it suffices to prove the estimates (4.4) and (4.5) for the case $g = 0$, $h = 0$.

Let $p(|z|)$ be the radial positive solution of the equation

$$-(-\Delta)^s p + 4q = 0 \text{ in } \mathbb{R}^n$$

with $q(|z|) = \frac{1}{1+|z|^{2s+\alpha}}$. Then by Riesz kernel, we get $p(z) \sim \frac{1}{1+|z|^\alpha}$. For a given sufficiently small $\delta > 0$, we have

$$-(-\Delta)^s p + \frac{\delta}{1+|z|^{2s}} p + 2q \leq 0 \text{ in } \mathbb{R}^n.$$

Thus $\bar{p}(x) := \sum_{j=1}^k p\left(\frac{x-\xi_j}{\mu_j}\right)$ satisfies

$$-(-\Delta)^s \bar{p} + \left(\sum_{j=1}^k \mu_j^{-2s} \frac{\delta}{1+|\frac{x-\xi_j}{\mu_j}|^{2s}} \right) \bar{p} + \frac{3}{2} \bar{q} \leq 0 \text{ in } \mathbb{R}^n$$

with $\bar{q} := \sum_{j=1}^k \mu_j^{-2s} q\left(\frac{x-\xi_j}{\mu_j}\right)$. From the definition of $V_{\mu,\xi}$, we have

$$|V_{\mu,\xi}| \lesssim \sum_{j=1}^k \mu_j^{-2s} \frac{R^{-2s}}{1+|y_j|^{2s}}. \quad (4.7)$$

For a given number $\beta > 0$, it is easy to see that $\bar{\psi}(x, t) = 2t^{-\beta} \bar{p}$ is a positive supersolution to

$$\partial_t \bar{\psi} = -(-\Delta)^s \bar{\psi} + V_{\mu,\xi} \bar{\psi} + t^{-\beta} \bar{q},$$

i.e.,

$$\partial_t \bar{\psi} \geq -(-\Delta)^s \bar{\psi} + V_{\mu,\xi} \bar{\psi} + t^{-\beta} \bar{q}$$

for $t > t_0$ and t_0 is sufficiently large. Therefore, one has

$$|\psi(x, t)| \lesssim t^{-\beta} \|f\|_{*,\beta,2s+\alpha} \sum_{j=1}^k \frac{1}{1+|y_j|^\alpha}. \quad (4.8)$$

and (4.4) is proved.

To prove (4.5), let

$$\psi(x, t) := \tilde{\psi}\left(\frac{x-\xi_j}{\mu_j}, \tau(t)\right)$$

where $\dot{\tau}(t) = \mu_j^{-2s}(t)$, namely $\tau(t) \sim t^{\frac{n-2s}{n-4s}}$. Without loss of generality, we assume $\tau(t_0) \geq 2$ by fixing t_0 . Then $\tilde{\psi}$ satisfies

$$\partial_\tau \tilde{\psi} = -(-\Delta)^s \tilde{\psi} + a(z, t) \cdot \nabla_z \tilde{\psi} + b(z, t) \tilde{\psi} + \tilde{f}(z, \tau) \quad (4.9)$$

for $|z| \leq \delta \mu_0^{-1}$ and $\tilde{f}(z, \tau) = \mu_j^{2s} f(\xi_j + \mu_j z, t(\tau))$. The uniformly small coefficients $a(z, t)$ and $b(z, t)$ in (4.9) are given by

$$a(z, t) := \mu_j^{2s-1} \dot{\mu}_j z + \mu_j^{2s-1} \dot{\xi}_j, \quad b(z, t) = \mu_j^{2s} V_{\mu,\xi}(\xi_j + \mu_j z).$$

Then from assumption (4.3) and (4.8) we have

$$|\tilde{f}(z, \tau)| \lesssim t(\tau)^{-\beta} \frac{\|f\|_{*,\beta,2s+\alpha}}{1+|z|^{2s+\alpha}}$$

and

$$|\tilde{\psi}(z, \tau)| \lesssim t(\tau)^{-\beta} \frac{\|f\|_{*,\beta,2s+\alpha}}{1+|z|^\alpha}.$$

Now fix $0 < \eta < 1$, from the regularity estimates for parabolic integro-differential equations (see [41]), for $\tau_1 \geq \tau(t_0) + 2$, we have

$$\begin{aligned} [\tilde{\psi}(\cdot, \tau_1)]_{\eta, B_{10}(0)} &\lesssim \|\tilde{\psi}\|_{L^\infty} + \|\tilde{f}\|_{L^\infty} \\ &\lesssim t(\tau_1 - 1)^{-\beta} \|f\|_{*, \beta, 2s+\alpha} \\ &\lesssim t(\tau_1)^{-\beta} \|f\|_{*, \beta, 2s+\alpha}. \end{aligned}$$

Therefore, choosing an appropriate constant c_n such that for any $t \geq c_n t_0$ we have

$$(R\mu_j)^\eta [\psi(\cdot, t)]_{\eta, B_{10\mu_j}(\xi_j)} \lesssim t^{-\beta} \|f\|_{*, \beta, 2s+\alpha}. \quad (4.10)$$

By the same token, the estimate (4.10) also holds for $t_0 \leq t \leq c_n t_0$. Thus, (4.5) holds for any $t \geq t_0$. The proof is completed. \square

4.2. Solvability of the outer problem. Now we fix σ satisfying

$$0 < \sigma < \bar{\sigma} \text{ where } \bar{\sigma} \leq \frac{n-2s}{2s} \bar{\sigma}_j b_j^{2-2s}, \quad j = 1, \dots, k, \quad (4.11)$$

and $\bar{\sigma}_j$ and b_j are defined in (2.28) and (2.19) respectively. Given $h(t) : (t_0, \infty) \rightarrow \mathbb{R}^k$ and $\delta > 0$, we define the weighted L^∞ norm as

$$\|h\|_\delta := \|\mu_0(t)^{-\delta} h(t)\|_{L^\infty(t_0, \infty)}.$$

In the rest of this paper, we always assume that a is a positive constant satisfying $a > 2s$ and $a - 2s$ is sufficiently small. We also assume the parameters $\lambda, \xi, \dot{\lambda}, \dot{\xi}$ satisfy the following two constraints,

$$\|\dot{\lambda}(t)\|_{n-4s+1+\sigma} + \|\dot{\xi}(t)\|_{n-4s+1+\sigma} \leq \frac{c}{R^{a-2s}}, \quad (4.12)$$

$$\|\lambda(t)\|_{1+\sigma} + \|\xi(t) - q\|_{1+\sigma} \leq \frac{c}{R^{a-2s}}, \quad (4.13)$$

where c is a positive constant independent of t, t_0 and R .

Denote

$$\|\phi\|_{n-2s+\sigma, a} = \max_{j=1, \dots, k} \|\phi_j\|_{n-2s+\sigma, a},$$

where $\|\phi_j\|_{n-2s+\sigma, a}$ is defined as the least number M such that

$$(1 + |y|) |\nabla_y \phi_j(y, t)| \chi_{B_{2R}(0)}(y) + |\phi_j(y, t)| \leq M \frac{\mu_0^{n-2s+\sigma}}{1 + |y|^a}, \quad j = 1, \dots, k \quad (4.14)$$

holds. Suppose $\phi = (\phi_1, \dots, \phi_k)$ satisfies

$$\|\phi\|_{n-2s+\sigma, a} \leq ct_0^{-\varepsilon} \quad (4.15)$$

for some small $\varepsilon > 0$. Then we have the following proposition.

Proposition 4.1. *Assume $\lambda, \xi, \dot{\lambda}, \dot{\xi}$ satisfy (4.12) and (4.13), $\phi = (\phi_1, \dots, \phi_k)$ satisfies (4.15), $\psi_0 \in C^2(\mathbb{R}^n)$ and we have*

$$\|\psi_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \psi_0\|_{L^\infty(\mathbb{R}^n)} \leq \frac{t_0^{-\varepsilon}}{R^{a-2s}}.$$

Then there exists t_0 sufficiently large such that the outer problem (4.1) has a unique solution $\psi = \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$. Moreover, there exists σ satisfying (4.11) and $\varepsilon > 0$ small such that, for $y_j = \frac{x - \xi_j}{\mu_{0j}}$,

$$|\psi(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2} + \sigma}(t)}{1 + |y_j|^{a-2s}} + e^{-\delta(t-t_0)} \|\psi_0\|_{L^\infty(\mathbb{R}^n)}, \quad (4.16)$$

and

$$[\psi(x, t)]_{\eta, B_{\mu_j R}(\xi_j)} \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-\eta} \mu_0^{\frac{n-2s}{2} + \sigma}(t)}{1 + |y_j|^{a-2s+\eta}} \text{ for } |y_j| \leq 2R, \quad (4.17)$$

where R, ρ are defined in (3.1) and $\eta \in (0, 1)$.

Proposition 4.1 states that for any small initial conditions ψ_0 , a solution ψ to (4.1) exists. Moreover, it clarifies the dependence of $\Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$ in the parameters $\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi$, which is proved by estimating, for example,

$$\partial_\phi \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi][\bar{\phi}] = \partial_s \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi + s\bar{\phi}]|_{s=0}$$

as a linear operator between parameter Banach spaces. For simplicity, we denote the above operator by $\partial_\phi \Psi[\bar{\phi}]$. Similarly, we have $\partial_\lambda \Psi[\bar{\lambda}], \partial_\xi \Psi[\bar{\xi}], \partial_{\dot{\lambda}} \Psi[\bar{\dot{\lambda}}], \partial_{\dot{\xi}} \Psi[\bar{\dot{\xi}}]$.

Proof. Lemma 4.1 defines a linear operator T which associates the solution $\psi = T(f, g, h)$ of problem (4.2) to any given functions $f(x, t), g(x, t)$ and $h(x)$. Denote $\psi_1(x, t) := T(0, -u_{\mu, \xi}^*, \psi_0)$. From (2.23), (2.4) and (2.22), for any $x \in \mathbb{R}^n \setminus \Omega$ we have

$$|u_{\mu, \xi}^*(x, t)| \lesssim \mu_0^{\frac{n+2s}{2}}(t). \quad (4.18)$$

By Lemma 4.1, we have

$$|\psi_1| \lesssim e^{-\delta(t-t_0)} \|\psi_0\|_{L^\infty(\mathbb{R}^n)} + t^{-\beta} \mu_0(t_0)^{2s-\sigma} \text{ where } \beta = \frac{n-2s}{2(n-4s)} + \frac{\sigma}{n-4s}.$$

Therefore, the function $\psi + \psi_1$ is a solution to (4.1) if ψ is a fixed point for the operator

$$\mathcal{A}(\psi) := T(f(\psi), 0, 0),$$

where

$$f(\psi) = \sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} + \tilde{N}_{\mu, \xi}(\tilde{\phi}) + S_{out}. \quad (4.19)$$

By the Contraction Mapping Theorem, we will prove the existence of a fixed point ψ for \mathcal{A} in the following function space

$$\|\psi\|_{**,\beta,a} \text{ is bounded with } \beta = \frac{n-2s}{2(n-4s)} + \frac{\sigma}{n-4s}.$$

Here $\|\psi\|_{**,\beta,a}$ is the least $M > 0$ such that the following inequality holds

$$|\psi(x, t)| \leq M \sum_{j=1}^k \frac{t^{-\beta}}{1 + |y_j|^{a-2s}}, \quad y_j = \frac{|x - \xi_j|}{\mu_j}.$$

As a first step, we establish the following estimates.

(1) Estimate for $S_{out}(x, t)$:

$$|S_{out}(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2} + \sigma}(t)}{1 + |y_j|^a}. \quad (4.20)$$

(2) Estimate for $\sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\}$:

$$\left| \sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \right| \quad (4.21)$$

$$\lesssim \frac{1}{R^{a-2s}} \|\phi\|_{n-2s+\sigma, a} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1+|y_j|^a}.$$

(3) Estimate for $\tilde{N}_{\mu, \xi}(\tilde{\phi})$:

$$\tilde{N}_{\mu, \xi}(\tilde{\phi}) \lesssim \begin{cases} t_0^{-\varepsilon} (\|\phi\|_{n-2s+\sigma, a}^2 + \|\psi\|_{**, \beta, a}^2) \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1+|y_j|^a}, & \text{when } 6s \geq n, \\ t_0^{-\varepsilon} (\|\phi\|_{n-2s+\sigma, a}^p + \|\psi\|_{**, \beta, a}^p) \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1+|y_j|^a}, & \text{when } 6s < n. \end{cases} \quad (4.22)$$

Proof of (4.20). Recall from (3.5) that

$$S_{out} = S_{\mu, \xi}^{(2)} + \sum_{j=1}^k (1 - \eta_{j,R}) S_{\mu, \xi, j}.$$

By (2.24) and Lemma 2.2, in the region $|x - q_j| > \delta$ with $\delta > 0$ small, S_{out} can be estimated for all j as

$$|S_{out}(x, t)| \lesssim \mu_0^{\frac{n-2s}{2}} (\mu_0^{2s} + \mu_0^{n-4s}) \lesssim \mu_0^{\min(n-4s, 2s) - (a-2s) - \sigma} (t_0) \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1+|y_j|^a}. \quad (4.23)$$

Now we consider the region $|x - q_j| \leq \delta$ with $\delta > 0$ small, where $j \in \{1, \dots, k\}$ is fixed. Lemma 2.2 implies that

$$\left| S_{\mu, \xi}^{(2)}(x, t) \right| \lesssim \mu_0^{-\frac{n+2s}{2}} \frac{\mu_0^{n-2s+2}}{1+|y_j|^{4s-2}} \lesssim \mu_0^{2s - (a-2s) - \sigma} (t_0) \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1+|y_j|^a}. \quad (4.24)$$

From the definition of $\eta_{j,R}$, $(1 - \eta_{j,R}) \neq 0$ if $|x - \xi_j| > \mu_0 R$. Therefore, in the region $|x - q_j| < \delta$,

$$|(1 - \eta_{j,R}) S_{\mu, \xi, j}| \lesssim \left(\frac{1}{R^{n-2s-a}} + \frac{1}{R^{4s-a}} \right) \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1+|y_j|^a}. \quad (4.25)$$

Here we have used the decaying assumptions (4.12) and (4.13) for λ and ξ , respectively. Thus, (4.20) is valid.

Proof of (4.21). First, we consider $[-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j]$ for j fixed. Recall that

$$\tilde{\phi}_j(x, t) := \mu_{0j}^{-\frac{n-2s}{2}} \phi_j \left(\frac{x - \xi_j}{\mu_{0j}}, t \right).$$

From the assumptions (4.14) and (4.15), we obtain

$$\begin{aligned}
 & \left| \left([-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] \right) (x, t) \right| \\
 & \lesssim \left[\int_{\mathbb{R}^n} \left(\frac{\eta_{j,R}(x) - \eta_{j,R}(y)}{|x-y|^{\frac{n}{2}+s}} \right)^2 dy \right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} \left(\frac{\tilde{\phi}_j(x) - \tilde{\phi}_j(y)}{|x-y|^{\frac{n}{2}+s}} \right)^2 dy \right]^{\frac{1}{2}} \\
 & \lesssim \frac{1}{R^s \mu_{0j}^s} \left[\int_{\mathbb{R}^n} \left(\frac{\eta(|\frac{x-\xi_j}{R\mu_{0j}}|) - \eta(|\frac{y-\xi_j}{R\mu_{0j}}|)}{|\frac{x-y}{R\mu_{0j}}|^{\frac{n}{2}+s}} \right)^2 d\left(\frac{y-\xi_j}{R\mu_{0j}}\right) \right]^{\frac{1}{2}} \\
 & \quad \times \frac{\mu_0^{-\frac{n-2s}{2}}}{\mu_{0j}^s} \left[\int_{\mathbb{R}^n} \left(\frac{\phi_j(\frac{x-\xi_j}{\mu_{0j}}, t) - \phi_j(\frac{y-\xi_j}{\mu_{0j}}, t)}{|\frac{x-y}{\mu_{0j}}|^{\frac{n}{2}+s}} \right)^2 d\left(\frac{y-\xi_j}{\mu_{0j}}\right) \right]^{\frac{1}{2}} \\
 & \lesssim \frac{1}{R^s \mu_{0j}^{2s}} \left[\int_{\mathbb{R}^n} \left(\frac{\eta(|\frac{x-\xi_j}{R\mu_{0j}}|) - \eta(|\frac{y-\xi_j}{R\mu_{0j}}|)}{|\frac{x-y}{R\mu_{0j}}|^{\frac{n}{2}+s}} \right)^2 d\left(\frac{y-\xi_j}{R\mu_{0j}}\right) \right]^{\frac{1}{2}} \frac{\mu_0^{\frac{n-2s}{2}+\sigma}}{(1+|y_j|^{s+a})} \|\phi\|_{n-2s+\sigma, a} \\
 & \lesssim \frac{1}{R^{a-2s}} \|\phi\|_{n-2s+\sigma, a} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1+|y_j|^a}.
 \end{aligned} \tag{4.26}$$

Now let us consider the second term $\tilde{\phi}_j(-(-\Delta)^s - \partial_t)\eta_{j,R}$. From direct computations, we have

$$\begin{aligned}
 \left| \tilde{\phi}_j(-(-\Delta)^s - \partial_t)\eta_{j,R} \right| & \lesssim \frac{\left| -(-\Delta)^s \eta\left(\left|\frac{x-\xi_j}{R\mu_{0j}}\right|\right) \right|}{R^{2s} \mu_{0j}^{2s}} \mu_0^{-\frac{n-2s}{2}} |\phi_j| \\
 & \quad + \left| \eta' \left(\left| \frac{x-\xi_j}{R\mu_{0j}} \right| \right) \left(\frac{|x-\xi_j|}{R\mu_0^2} \mu_0 + \frac{1}{R\mu_0} \dot{\xi} \right) \right| \mu_0^{-\frac{n-2s}{2}} |\phi_j|.
 \end{aligned} \tag{4.27}$$

For the first term in the right hand side of (4.27), by the definition of $\tilde{\phi}_j$, we obtain

$$\begin{aligned}
 \frac{\left| -(-\Delta)^s \eta\left(\left|\frac{x-\xi_j}{R\mu_{0j}}\right|\right) \right|}{R^{2s} \mu_{0j}^{2s}} \mu_0^{-\frac{n-2s}{2}} |\phi_j| & \lesssim \frac{\left| -(-\Delta)^s \eta\left(\left|\frac{x-\xi_j}{R\mu_{0j}}\right|\right) \right|}{R^{2s} \mu_{0j}^{2s}} \frac{\mu_0^{\frac{n-2s}{2}+\sigma}}{(1+|y_j|^a)} \|\phi\|_{n-2s+\sigma, a} \\
 & \lesssim \frac{1}{R^{a-2s}} \|\phi\|_{n-2s+\sigma, a} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1+|y_j|^a},
 \end{aligned} \tag{4.28}$$

where we have used the fact that $\left| -(-\Delta)^s \eta\left(\left|\frac{x-\xi_j}{R\mu_{0j}}\right|\right) \right| \sim \frac{1}{1+|\frac{y_j}{R}|^{2s}}$. From (2.18) and (4.12), the second term in the right hand side of (4.27) can be estimated as

$$\left| \eta' \left(\left| \frac{x-\xi_j}{R\mu_{0j}} \right| \right) \left(\frac{|x-\xi_j| \mu_0 + \mu_0 \dot{\xi}}{R\mu_0^2} \right) \right| \mu_0^{-\frac{n-2s}{2}} |\phi_j|$$

$$\begin{aligned}
&\lesssim \frac{\left| \eta' \left(\left| \frac{x-\xi_j}{R\mu_{0j}} \right| \right) \right|}{R^{2s}\mu_{0j}^{2s}} (\mu_0^{n-2s} R^{2s} + \mu_0^{n-2s+\sigma} R^{2s-1}) \mu_0^{-\frac{n-2s}{2}} |\phi_j| \\
&\lesssim \frac{1}{R^{a-2s}} \|\phi\|_{n-2s+\sigma, a} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1+|y_j|^a}.
\end{aligned} \tag{4.29}$$

From (4.26)-(4.29), we get (4.21).

Proof of (4.22). Since $p-2 \geq 0$ gives $6s \geq n$, we have

$$\begin{aligned}
\tilde{N}_{\mu, \xi}(\psi + \psi_1 + \sum_{j=1}^k \eta_{j,R} \tilde{\phi}_j) &\lesssim \\
&\begin{cases} (u_{\mu, \xi}^*)^{p-2} \left[|\psi|^2 + |\psi_1|^2 + \sum_{j=1}^k |\eta_{j,R} \tilde{\phi}_j|^2 \right], & \text{when } 6s \geq n, \\ |\psi|^p + |\psi_1|^p + \sum_{j=1}^k |\eta_{j,R} \tilde{\phi}_j|^p, & \text{when } 6s < n. \end{cases}
\end{aligned} \tag{4.30}$$

When $6s \geq n$, we have

$$\begin{aligned}
\left| (u_{\mu, \xi}^*)^{p-2} (\eta_{j,R} \tilde{\phi}_j)^2 \right| &\lesssim \frac{\mu_0^{\frac{3n}{2}-5s+2\sigma}}{1+|y_j|^{2a}} \|\phi\|_{n-2s+\sigma, a}^2 \\
&\lesssim \mu_0^{n-2s+\sigma} R^{a-2s} \|\phi\|_{n-2s+\sigma, a}^2 \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1+|y_j|^a}
\end{aligned}$$

and

$$\begin{aligned}
\left| (u_{\mu, \xi}^*)^{p-2} \psi^2 \right| &\lesssim \mu_0^{-\frac{6s-n}{2}} \frac{t^{-2\beta}}{1+|y_j|^{2(a-2s)}} \|\psi\|_{**, \beta, a}^2 \\
&\lesssim R^{a-2s} \mu_0^{n-4s+\sigma+a-2s} \|\psi\|_{**, \beta, a}^2 \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} t^{-\beta}}{1+|y_j|^a}.
\end{aligned}$$

When $6s < n$, we have

$$\begin{aligned}
\left| \eta_{j,R} \tilde{\phi}_j \right|^p &\lesssim \frac{\mu_0^{(\frac{n-2s}{2}+\sigma)p}}{1+|y_j|^{ap}} \|\phi\|_{n-2s+\sigma, a}^p \\
&\lesssim \mu_0^{2s+(p-1)\sigma} R^{a-2s} \mu_0^{2s} \|\phi\|_{n-2s+\sigma, a}^p \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1+|y_j|^a},
\end{aligned}$$

and

$$\begin{aligned}
|\psi|^p &\lesssim \frac{t^{-p\beta}}{1+|y_j|^{p(a-2s)}} \|\psi\|_{**, \beta, a}^p \\
&\lesssim \mu^{4s(1+\frac{\sigma}{n-2s})+p(a-2s)-a} R^{a-2s} \|\psi\|_{**, \beta, a}^p \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1+|y_j|^a}.
\end{aligned}$$

The estimates for ψ_1 are similar. Hence we have (4.22).

Now we apply the Contraction Mapping Theorem to prove the existence of a fixed point ψ for \mathcal{A} . First, set

$$\mathcal{B} = \left\{ \psi : \|\psi\|_{**,\beta,a} \leq M \frac{t_0^{-\varepsilon}}{R^{a-2s}} \right\}$$

with $\beta = \frac{n-2s}{2(n-4s)} + \frac{\sigma}{n-4s}$ and a, ε are fixed as above. Here the positive large constant M is independent of t and t_0 . For any $\psi \in \mathcal{B}$, $\mathcal{A}(\psi) \in \mathcal{B}$ as a consequence of (4.19) and the estimates (4.20)-(4.22). We claim that for any $\psi_1, \psi_2 \in \mathcal{B}$,

$$\|\mathcal{A}(\psi^{(1)}) - \mathcal{A}(\psi^{(2)})\|_{**,\beta,a} \leq C \|\psi^{(1)} - \psi^{(2)}\|_{**,\beta,a},$$

where $C < 1$ is a constant depending on t_0 which is chosen sufficiently large. Indeed,

$$\mathcal{A}(\psi^{(1)}) - \mathcal{A}(\psi^{(2)}) = T \left(\tilde{N}_{\mu,\xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu,\xi}(\psi^{(2)} + \psi_1 + \phi^{in}), 0, 0 \right),$$

where

$$\begin{aligned} & \tilde{N}_{\mu,\xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu,\xi}(\psi^{(2)} + \psi_1 + \phi^{in}) = \\ & \left(u_{\mu,\xi}^* + \psi^{(1)} + \psi_1 + \phi^{in} \right)^p - \left(u_{\mu,\xi}^* + \psi^{(2)} + \psi_1 + \phi^{in} \right)^p - p(u_{\mu,\xi}^*)^{p-1} \left[\psi^{(1)} - \psi^{(2)} \right]. \end{aligned}$$

Similar to (4.30), we have

$$\begin{aligned} \left| \tilde{N}_{\mu,\xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu,\xi}(\psi^{(2)} + \psi_1 + \phi^{in}) \right| & \lesssim \\ & \begin{cases} (u_{\mu,\xi}^*)^{p-2} |\phi^{in}| |\psi^{(1)} - \psi^{(2)}|, & \text{when } 6s \geq n, \\ |\phi^{in}|^{p-1} |\psi^{(1)} - \psi^{(2)}|, & \text{when } 6s < n. \end{cases} \end{aligned}$$

When $6s \geq n$,

$$\begin{aligned} & \left| \tilde{N}_{\mu,\xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu,\xi}(\psi^{(2)} + \psi_1 + \phi^{in}) \right| \\ & \lesssim \|\phi\|_{n-2s+\sigma,a} \|\psi^{(1)} - \psi^{(2)}\|_{**,\beta,a} R^{a-2s} \mu_0^{\frac{n}{2}+s+\sigma} (t_0) \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} t^{-\beta}}{1 + |y_j|^a}, \end{aligned}$$

while in the case of $6s < n$,

$$\begin{aligned} & \left| \tilde{N}_{\mu,\xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu,\xi}(\psi^{(2)} + \psi_1 + \phi^{in}) \right| \\ & \lesssim \|\phi\|_{n-2s+\sigma,a}^{p-1} \|\psi^{(1)} - \psi^{(2)}\|_{**,\beta,a} R^{a-2s} \mu_0^{2s+\frac{4s(\sigma+a)}{n-2s}} (t_0) \frac{1}{R^{a-2s}} \sum_{j=1}^k \frac{\mu_j^{-2s} t^{-\beta}}{1 + |y_j|^a}. \end{aligned}$$

Hence there exists a choice of R in the form (3.1) such that

$$\|\mathcal{A}(\psi^{(1)}) - \mathcal{A}(\psi^{(2)})\|_{**,\beta,a} \leq C \|\psi^{(1)} - \psi^{(2)}\|_{**,\beta,a}$$

holds with $C < 1$, provided t_0 is sufficiently large. Therefore, if t_0 is fixed sufficiently large, \mathcal{A} is a contraction map in \mathcal{B} . The validity of (4.17) follows directly from (4.5). The proof is completed. \square

4.3. Properties of the solution ψ .

Proposition 4.2. *Under the assumptions in Proposition 4.1, Ψ depends smoothly on the parameters $\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi$, for $y_j = \frac{x - \xi_j}{\mu_0}$, we have*

$$|\partial_\lambda \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi][\bar{\lambda}](x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}(t)\|_{1+\sigma} \left(\sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2}-1}(t)}{1 + |y_j|^{a-2s}} \right), \quad (4.31)$$

$$|\partial_\xi \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi][\bar{\xi}](x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\xi}(t)\|_{1+\sigma} \left(\sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2}-1}(t)}{1 + |y_j|^{a-2s}} \right), \quad (4.32)$$

$$|\partial_{\dot{\xi}} \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi][\dot{\bar{\xi}}](x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\dot{\bar{\xi}}(t)\|_{n-4s+1+\sigma} \left(\sum_{j=1}^k \frac{\mu_0^{\frac{-n-6s}{2}-1+\sigma}(t)}{1 + |y_j|^{a-2s}} \right), \quad (4.33)$$

$$|\partial_{\dot{\lambda}} \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi][\dot{\bar{\lambda}}](x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\dot{\bar{\lambda}}(t)\|_{n-4s+1+\sigma} \left(\sum_{j=1}^k \frac{\mu_0^{\frac{-n-6s}{2}-1+\sigma}(t)}{1 + |y_j|^{a-2s}} \right), \quad (4.34)$$

$$|\partial_\phi \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi][\bar{\phi}](x, t)| \lesssim \frac{1}{R^{a-2s}} \|\bar{\phi}(t)\|_{n-2s+\sigma, a} \left(\sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2}+\sigma}(t)}{1 + |y_j|^{a-2s}} \right). \quad (4.35)$$

Proof. Step 1. Proof of (4.31) and (4.32).

We fix $j = 1$. $\Psi[\lambda_1]$ is a solution to problem (4.1) for all λ_1 satisfying (4.13). Differentiating problem (4.1) with respect to λ_1 gives us a nonlinear equation. From the Implicit Function Theorem, the solutions are given by $\partial_{\lambda_1} \Psi[\bar{\lambda}_1](x, t)$. Decompose $\partial_{\lambda_1} \Psi[\bar{\lambda}_1](x, t) = Z_1 + Z$ with $Z_1 = T(0, -(\partial_{\lambda_1} u_{\mu, \xi}^*)[\bar{\lambda}_1], 0)$, where T is defined in Lemma 4.1. Then Z is a solution of the following nonlinear problem

$$\begin{cases} \partial_t Z = -(-\Delta)^s Z + V_{\mu, \xi} Z + (\partial_{\lambda_1} V_{\mu, \xi})[\bar{\lambda}_1] \psi + \partial_{\lambda_1} [\tilde{N}_{\mu, \xi}(\psi + \phi^{in})][\bar{\lambda}_1] \\ \quad + \partial_{\lambda_1} S_{out}[\bar{\lambda}_1] \quad \text{in } \Omega \times (t_0, \infty), \\ Z = 0 \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\ Z(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^n. \end{cases} \quad (4.36)$$

By definition, $\sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\}$ is independent of λ_1 . Then for any $x \in \mathbb{R}^n \setminus \Omega$,

$$|\partial_{\lambda_1} u_{\mu, \xi}^*(x, t)| \lesssim \mu_0^{\frac{n-2s}{2}-1}(t) |\bar{\lambda}_1(t)|. \quad (4.37)$$

From (4.37) and Lemma 4.1, we obtain

$$|Z_1(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \left(\sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2}-1}(t)}{1 + |y_j|^{a-2s}} \right).$$

For problem (4.36), we compute

$$\begin{aligned} \partial_{\lambda_1} [\tilde{N}_{\mu, \xi}(\psi + \phi^{in})][\bar{\lambda}_1] &= p [(u_{\mu, \xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu, \xi}^*)^{p-1}] (Z + Z_1) \\ &\quad + p(p-1)(u_{\mu, \xi}^*)^{p-2} (\psi + \phi^{in}) \partial_{\lambda_1} u_{\mu, \xi}^*[\bar{\lambda}_1]. \end{aligned}$$

Therefore, Z is a fixed point of the operator

$$\mathcal{A}_1(Z) = T(f + p [(u_{\mu,\xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu,\xi}^*)^{p-1}] Z, 0, 0), \quad (4.38)$$

where

$$\begin{aligned} f = & \partial_{\lambda_1} S_{out}[\bar{\lambda}_1] + (\partial_{\lambda_1} V_{\mu,\xi})[\bar{\lambda}_1]\psi + p [(u_{\mu,\xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu,\xi}^*)^{p-1}] Z_1 \\ & + p(p-1)(u_{\mu,\xi}^*)^{p-2}(\psi + \phi^{in})\partial_{\lambda_1} u_{\mu,\xi}^*[\bar{\lambda}_1]. \end{aligned} \quad (4.39)$$

We claim that

$$|f(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}-1+\sigma}}{1 + |y_j|^a}. \quad (4.40)$$

To prove (4.40), we first estimate $\partial_{\lambda_1} S_{out}[\bar{\lambda}_1]$. In the region $|x - q_i| > \delta$ ($i = 1, \dots, k$), we have the following estimate for $\partial_{\lambda_1} S(u_{\mu,\xi}^*)$ by (2.24), (4.12) and (4.13)

$$\partial_{\lambda_1} S(u_{\mu,\xi}^*)[\bar{\lambda}_1](x, t) = \mu_0^{\frac{n-2s}{2}-1} f(x, \mu_0^{-1}\mu, \xi) \bar{\lambda}_1(t),$$

where the smooth and bounded function f depends on $(x, \mu_0^{-1}\mu, \xi)$. Now we fix j and consider the region $|x - q_j| \leq \delta$. From (2.26), we have

$$\partial_{\lambda_1} S(u_{\mu,\xi}^*)[\bar{\lambda}_1](x, t) = \partial_{\lambda_1} S(u_{\mu,\xi})[\bar{\lambda}_1](x, t)(1 + \mu_0 f(x, \mu_0^{-1}\mu, \xi, t)),$$

where the smooth and bounded function f depends on $(x, \mu_0^{-1}\mu, \xi, t)$. Differentiating (2.5) with respect to λ_1 , we obtain

$$\begin{aligned} \partial_{\lambda_1} S(u_{\mu,\xi})[\bar{\lambda}_1](x, t) = & - \left(\frac{n-2s}{2} + 1 \right) \mu_1^{-\frac{n-2s}{2}-2} \left[\dot{\mu}_1 Z_{n+1}(y_1) + \dot{\xi}_1 \cdot \nabla U(y_1) \right. \\ & \left. - \frac{1}{\frac{n-2s}{2} + 1} \frac{n-2s}{2} \left(\frac{n-2s}{2} - 1 \right) \mu_1^{n-2s} \dot{\mu}_1 H(x, q_i) \right] \bar{\lambda}_1(t) \\ & - \mu_i^{-\frac{n-2s}{2}-1} \left[\dot{\xi}_1 D^2 U(y_1) + \dot{\mu}_1 \nabla Z_{n+1}(y_1) \right] \cdot \frac{x - \xi_1}{\mu_1^2} \bar{\lambda}_1(t) \\ & + p \left(\sum_{i=1}^k \mu_i^{-\frac{n-2s}{2}} U(y_i) - \mu_i^{-\frac{n-2s}{2}} H(x, q_i) \right)^{p-1} \\ & \quad \times \partial_{\lambda_1} \left[\mu_1^{-\frac{n-2s}{2}} U(y_1) - \mu_1^{-\frac{n-2s}{2}} H(x, q_1) \right] \bar{\lambda}_1(t) \\ & - p \left(\mu_1^{-\frac{n-2s}{2}} U(y_1) \right)^{p-1} \partial_{\lambda_1} \left[\mu_1^{-\frac{n-2s}{2}} U(y_1) \right] \bar{\lambda}_1(t). \end{aligned}$$

From (4.12) and (4.13), we have

$$|\partial_{\lambda_1} S(u_{\mu,\xi})[\bar{\lambda}_1](x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}-1}}{1 + |y_j|^a}. \quad (4.41)$$

Therefore, by the definition of S_{out} together with (4.41), we obtain

$$|\partial_{\lambda_1} S_{out}[\bar{\lambda}_1](x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}-1}}{1 + |y_j|^a}.$$

Next, we estimate the remainders in f . Direct computations imply that

$$\begin{aligned} (\partial_{\lambda_1} V_{\mu, \xi})[\bar{\lambda}_1](x, t) = & p(p-1) \left[(u_{\mu, \xi}^*)^{p-2} \partial_{\lambda_1} u_{\mu, \xi}^*[\bar{\lambda}_1] \right. \\ & \left. - \eta_{1,R} (\mu_1^{-\frac{n-2s}{2}} U(y_1))^{p-2} \partial_{\lambda_1} (\mu_1^{-\frac{n-2s}{2}} U(y_1))[\bar{\lambda}_1] \right]. \end{aligned}$$

Since $\left| \partial_{\lambda_1} \left(\mu_1^{-\frac{n-2s}{2}} U(y_1) \right) \right| \lesssim \mu_0^{-1} \left| \mu_1^{-\frac{n-2s}{2}} U(y_1) \right|$ and $\beta = \frac{n-2s}{2(n-4s)} + \frac{\sigma}{n-4s}$, we have

$$|(\partial_{\lambda_1} V_{\mu, \xi})[\bar{\lambda}_1] \psi(x, t)| \lesssim \|\psi\|_{**,\beta,a} \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}-1+\sigma}}{1+|y_j|^a}.$$

By the same token, we can deal with $p(p-1)(u_{\mu, \xi}^*)^{p-2}(\psi + \phi^{in})\partial_{\lambda_1} u_{\mu, \xi}^*[\bar{\lambda}_1]$ in (4.39) and obtain

$$\left| p(p-1)(u_{\mu, \xi}^*)^{p-2}(\psi + \phi^{in})\partial_{\lambda_1} u_{\mu, \xi}^*[\bar{\lambda}_1] \right| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}-1+\sigma}}{1+|y_j|^a}.$$

Analogously, we can estimate the last term $p \left[(u_{\mu, \xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu, \xi}^*)^{p-1} \right] Z_1$. Therefore, we conclude the validity of (4.40).

Now we consider the fixed point problem (4.38). Then the operator \mathcal{A}_1 has a fixed point in the set of functions satisfying

$$|Z(x, t)| \leq M \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2}-1}}{1+|y_j|^{a-2s}}$$

with the large constant M fixed. In fact, \mathcal{A}_1 is a contraction map when R is chosen properly large in terms of t_0 . Therefore, the estimate (4.31) for $\partial_{\lambda_1} \Psi[\bar{\lambda}_1]$ holds. The estimate (4.32) for $\partial_{\xi} \Psi[\bar{\xi}]$ can be verified in a similar way. Here we omit the details.

Step 2. Proof of (4.33) and (4.34).

We fix $j = 1$. From the discussions above, the function $\Psi[\dot{\lambda}_1]$ is a solution to (4.1) for all $\dot{\lambda}_1$ satisfying (4.13). Then we differentiate problem (4.1) with respect to $\dot{\lambda}_1$ and obtain a nonlinear equation. From the Implicit Function Theorem, the solutions are given by $\partial_{\dot{\lambda}_1} \Psi[\dot{\lambda}_1](x, t)$. Denote $Z(x, t) = \partial_{\dot{\lambda}_1} \Psi[\dot{\lambda}_1](x, t)$. Then Z is a solution to the following nonlinear problem

$$\begin{cases} \partial_t Z = \\ \quad - (-\Delta)^s Z + V_{\mu, \xi} Z + \partial_{\dot{\lambda}_1} \left[\tilde{N}_{\mu, \xi}(\psi + \phi^{in}) \right] [\dot{\lambda}_1] + \partial_{\dot{\lambda}_1} S_{out}[\dot{\lambda}_1] & \text{in } \Omega \times (t_0, \infty), \\ Z(x, t) = 0 & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\ Z(\cdot, t_0) = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

From the definition of $\tilde{N}_{\mu, \xi}(\psi + \phi^{in})$, we have

$$\partial_{\dot{\lambda}_1} \left[\tilde{N}_{\mu, \xi}(\psi + \phi^{in}) \right] [\dot{\lambda}_1] = p \left[(u_{\mu, \xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu, \xi}^*)^{p-1} \right] Z(x, t).$$

Therefore, Z is a fixed point for the operator

$$\mathcal{A}_1(Z) = T \left(\partial_{\dot{\lambda}_1} S_{out}[\dot{\lambda}_1] + p \left[(u_{\mu, \xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu, \xi}^*)^{p-1} \right] Z, 0, 0 \right). \quad (4.42)$$

Now we differentiate $S(u_{\mu,\xi}^*)$ with respect to $\dot{\lambda}_1$ in (2.25) directly and obtain

$$\begin{aligned} \partial_{\dot{\lambda}_1} S(u_{\mu,\xi}^*)[\dot{\lambda}_1](x, t) &= \mu_1^{-\frac{n-2s}{2}-1} \left[Z_{n+1}(y_1) + \frac{n-2s}{2} \mu_1^{n-2s} H(x, q_1) \right] \dot{\lambda}_1(t) \\ &\quad + \mu_j^{-\frac{n}{2}+s-1} \left[\frac{n-2s}{2} \Phi_1(y_1, t) + y_1 \cdot \nabla_y \Phi_1 \right] \dot{\lambda}_1(t). \end{aligned}$$

Hence

$$\left| \partial_{\dot{\lambda}_1} S(u_{\mu,\xi}^*)[\dot{\lambda}_1](x, t) \right| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\dot{\lambda}_1(t)\|_{n-4s+1+\sigma} \left(\sum_{j=1}^k \frac{\mu_j^{-2s}(t) \mu_0^{-\frac{n-6s}{2}-1}}{1 + |y_j|^a} \right).$$

Now we consider the fixed point problem (4.42). Similar to Step 1, \mathcal{A}_1 has a fixed point in the set of functions satisfying

$$|Z(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\dot{\lambda}_1(t)\|_{n-4s+1+\sigma} \sum_{j=1}^k \frac{\mu_0^{-\frac{n-6s}{2}-1}}{1 + |y_j|^{a-2s}}.$$

Thus estimate (4.33) holds.

On the other hand, observe that

$$\partial_{\dot{\xi}_1} S(u_{\mu,\xi}^*)[\dot{\xi}_1](x, t) = \mu_1^{-\frac{n-2s}{2}-1} [\nabla U(y_1) + \nabla \Phi_1(y_1, t)] \dot{\xi}_1(t). \quad (4.43)$$

From (4.43) we have

$$\left| \partial_{\dot{\xi}_1} S(u_{\mu,\xi}^*)[\dot{\xi}_1](x, t) \right| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\dot{\xi}_1(t)\|_{n-4s+1+\sigma} \left(\sum_{j=1}^k \frac{\mu_j^{-2s}(t) \mu_0^{-\frac{n-6s}{2}-1}}{1 + |y_j|^a} \right).$$

Therefore, we have (4.34).

Step 3. Proof of (4.35).

Define $Z(x, t) = \partial_\phi \psi[\bar{\phi}](x, t)$ with $\bar{\phi}$ satisfying (4.15). Therefore, Z is a solution to

$$\begin{cases} \partial_t Z = -(-\Delta)^s Z + V_{\mu,\xi} Z \\ \quad + \sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \hat{\phi}_j] + \hat{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \\ \quad + p [(u_{\mu,\xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu,\xi}^*)^{p-1}] \bar{\phi} \quad \text{in } \Omega \times (t_0, \infty), \\ Z = 0 \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\ Z(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^n, \end{cases}$$

where $\hat{\phi} = \mu_0^{-\frac{n-2s}{2}} \bar{\phi}_j \left(\frac{x - \xi_j}{\mu_{0j}}, t \right)$.

As in Step 1 and Step 2, we have

$$\begin{aligned} &\left| \sum_{j=1}^k \left\{ [-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \hat{\phi}_j] + \hat{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \right| \\ &\lesssim \frac{1}{R^{a-2s}} \|\bar{\phi}\|_{n-2s+\sigma, a} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2}+\sigma}}{1 + |y_j|^a} \end{aligned}$$

and

$$\begin{aligned} & \left| p \left[(u_{\mu, \xi}^* + \psi + \phi^{in})^{p-1} - (u_{\mu, \xi}^*)^{p-1} \right] \bar{\phi} \right| \\ & \lesssim \frac{1}{R^{a-2s}} \|\bar{\phi}\|_{n-2s+\sigma, a} \left[\|\psi\|_{**\beta, a}^{p-1} + \|\phi^{in}\|_{n-2s+\sigma, a}^{p-1} \right] \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{\frac{n-2s}{2} + \sigma}}{1 + |y_j|^a}. \end{aligned}$$

From Lemma 4.1, we conclude the validity of (4.35). \square

5. THE INNER PROBLEM

Substituting the solution $\psi = \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$ of the outer problem given by proposition 4.1 into the inner problem (3.8), the full problem is reduced to the following system

$$\mu_{0j}^{2s} \partial_t \phi_j = -(-\Delta)_y^s \phi_j + pU^{p-1}(y)\phi_j + H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t), \quad y \in \mathbb{R}^n, \quad t \geq t_0$$

for $j = 1, \dots, k$, where

$$\begin{aligned} H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi] := & \left\{ \mu_{0j}^{\frac{n+2s}{2}} S_{\mu, \xi, j}(\xi_j + \mu_{0j}y, t) + B_j[\phi_j] + B_j^0[\phi_j] \right. \\ & \left. + p\mu_{0j}^{\frac{n-2s}{2}} \frac{\mu_{0j}^{2s}}{\mu_j^{2s}} U^{p-1} \left(\frac{\mu_{0j}}{\mu_j} y \right) \psi(\xi_j + \mu_{0j}y, t) \right\} \chi_{B_{2R}(0)}(y) \end{aligned} \quad (5.2)$$

and $B_j[\phi_j]$ and $B_j^0[\phi_j]$ are defined in (3.9), (3.10) respectively.

After the change of variables

$$t = t(\tau), \quad \frac{dt}{d\tau} = \mu_{0j}^{2s}(t),$$

(5.1) is reduced to

$$\partial_\tau \phi_j = -(-\Delta)_y^s \phi_j + pU^{p-1}(y)\phi_j + H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)), \quad y \in \mathbb{R}^n, \quad \tau \geq \tau_0 \quad (5.3)$$

with τ_0 the unique positive number satisfying $t(\tau_0) = t_0$.

We will find a solution $\phi = (\phi_1, \dots, \phi_k)$ to the system

$$\begin{cases} \partial_\tau \phi_j = -(-\Delta)_y^s \phi_j + pU^{p-1}(y)\phi_j + H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)), & y \in \mathbb{R}^n, \quad \tau \geq \tau_0, \\ \phi_j(y, \tau_0) = e_{0j} Z_0(y), & y \in \mathbb{R}^n, \end{cases} \quad (5.4)$$

for a constant e_{0j} and all $j = 1, \dots, k$. Here Z_0 is a radially symmetric eigenfunction associated to the unique negative eigenvalue λ_0 of the eigenvalue problem

$$L_0(\phi) + \lambda\phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).$$

Note that λ_0 is simple and Z_0 satisfies

$$Z_0(y) \sim |y|^{-n-2s} \text{ as } |y| \rightarrow \infty,$$

see, for example, [30]. We will prove that (5.4) is solvable in the function space of those ϕ_j 's satisfying (4.15), provided ξ and λ are chosen so that $H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau))$ satisfy the orthogonality conditions

$$\int_{B_{2R}} H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)) Z_l(y) dy = 0, \quad (5.5)$$

for all $\tau \geq \tau_0$, $j = 1, \dots, k$ and $l = 1, 2, \dots, n+1$. We first develop a linear theory which is the context of the subsection 5.1.

5.1. The linear theory. In this subsection, for $R > 0$ fixed large, we find a solution to the nonlocal initial value problem

$$\begin{cases} \partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau), & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \phi(y, \tau_0) = e_0 Z_0(y), & y \in \mathbb{R}^n. \end{cases} \quad (5.6)$$

Let

$$\nu = 1 + \frac{\sigma}{n - 2s},$$

then $\mu_0^{n-2s+\sigma} \sim \tau^{-\nu}$. Define

$$\|h\|_{a,\nu,\eta} := \sup_{\tau > \tau_0} \sup_{y \in B_{2R}} \tau^\nu (1 + |y|^a) (|h(y, \tau)| + (1 + |y|^\eta) \chi_{B_{2R}(0)}(y) [h(\cdot, \tau)]_{\eta, B_1(0)}).$$

In the following, we always assume that $h = h(y, \tau)$ is a function defined in the whole space \mathbb{R}^n which is zero outside $B_{2R}(0)$ for all $\tau > \tau_0$. The main result in this subsection is the following.

Proposition 5.1. *Suppose $a \in (2s, n - 2s)$, $\nu > 0$, $\|h\|_{2s+a,\nu,\eta} < +\infty$ and*

$$\int_{B_{2R}} h(y, \tau) Z_j(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \infty), j = 1, \dots, n + 1.$$

For sufficiently large R , there exist $\phi = \phi[h](y, \tau)$ and $e_0 = e_0[h](\tau)$ ($\tau \in (\tau_0, +\infty)$, $y \in \mathbb{R}^n$) satisfying (5.6) and

$$(1 + |y|) |\nabla_y \phi(y, \tau)| \chi_{B_{2R}(0)}(y) + |\phi(y, \tau)| \lesssim \tau^{-\nu} (1 + |y|)^{-a} \|h\|_{2s+a,\nu,\eta}, \quad \tau \in (\tau_0, +\infty), y \in \mathbb{R}^n, \quad (5.7)$$

$$|e_0[h]| \lesssim \|h\|_{2s+a,\nu,\eta}. \quad (5.8)$$

Lemma 5.1. *Suppose $a \in (2s, n - 2s)$, $\nu > 0$, $\|h\|_{2s+a,\nu,\eta} < +\infty$ and*

$$\int_{\mathbb{R}^n} h(y, \tau) Z_j(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \infty), j = 1, \dots, n + 1.$$

For any sufficiently large $\tau_1 > 0$, the solution $(\phi(y, \tau), c(\tau))$ of the problem

$$\begin{cases} \partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau) - c(\tau) Z_0(y), & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \int_{\mathbb{R}^n} \phi(y, \tau) Z_0(y) dy = 0 & \text{for all } \tau \in (\tau_0, +\infty), \\ \phi(y, \tau_0) = 0, & y \in \mathbb{R}^n, \end{cases} \quad (5.9)$$

satisfies the estimates

$$\|\phi(y, \tau)\|_{a,\tau_1} \lesssim \|h\|_{2s+a,\tau_1} \quad (5.10)$$

and

$$|c(\tau)| \lesssim \tau^{-\nu} R^a \|h\|_{2s+a,\tau_1} \quad \text{for } \tau \in (\tau_0, \tau_1).$$

Here $\|h\|_{b,\tau_1} := \sup_{\tau \in (\tau_0, \tau_1)} \tau^\nu \|(1 + |y|^b)h\|_{L^\infty(\mathbb{R}^n)}$.

Proof. Note that (5.9) is equivalent to

$$\begin{cases} \partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau) - c(\tau) Z_0(y), & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \phi(y, \tau_0) = 0, & y \in \mathbb{R}^n \end{cases} \quad (5.11)$$

for $c(\tau)$ given by the relation

$$c(\tau) \int_{\mathbb{R}^n} |Z_0(y)|^2 dy = \int_{\mathbb{R}^n} h(y, \tau) Z_0(y) dy.$$

It is easy to see that

$$|c(\tau)| \lesssim \tau^{-\nu} R^a \|h\|_{2s+a, \tau_1} \quad (5.12)$$

holds for $\tau \in (\tau_0, \tau_1)$. So we only need to prove (5.10) for the solution ϕ of (5.11). Inspired by Lemma 4.5 of [20] and the linear theory of [43], we will use the blow-up argument.

First, we claim that, given $\tau_1 > \tau_0$, we have $\|\phi\|_{a, \tau_1} < +\infty$. Indeed, by the fractional parabolic theory (see [34]), given $R_0 > 0$ there is a $K = K(R_0, \tau_1)$ such that

$$|\phi(y, \tau)| \leq K \quad \text{in } B_{R_0}(0) \times (\tau_0, \tau_1].$$

Fix R_0 large and take K_1 sufficiently large, $K_1 \rho^{-a}$ is a supersolution for (5.11) when $\rho > R_0$. Hence $|\phi| \leq 2K_1 \rho^{-a}$ and $\|\phi\|_{a, \tau_1} < +\infty$ for any $\tau_1 > 0$. Next, we claim that the following identities hold,

$$\int_{\mathbb{R}^n} \phi(y, \tau) \cdot Z_j(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \tau_1), j = 0, 1, \dots, n+1. \quad (5.13)$$

Indeed, From the definition of $c(\tau)$, we have

$$\int_{\mathbb{R}^n} \phi(y, \tau) \cdot Z_0(y) dy = 0.$$

Testing (5.11) with $Z_j \eta$, where $\eta(y) = \eta_0(|y|/R_1)$, $j = 1, \dots, n+1$, R_1 is an arbitrary positive constant and the smooth cut-off function η_0 is defined as

$$\eta_0(r) = \begin{cases} 1, & \text{for } r < 1, \\ 0, & \text{for } r > 2, \end{cases}$$

we get

$$\int_{\mathbb{R}^n} \phi(\cdot, \tau) \cdot Z_j \eta = \int_0^\tau ds \int_{\mathbb{R}^n} (\phi(\cdot, s) \cdot L_0[\eta Z_j] + h Z_j \eta - c(s) Z_0 Z_j \eta).$$

Furthermore, it holds that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\phi \cdot L_0[\eta Z_j] + h Z_j \eta - c(s) Z_0 Z_j \eta \right) \\ &= \int_{\mathbb{R}^n} \phi \cdot \left(Z_j(-(-\Delta)^s) \eta + [-(-\Delta)^{\frac{s}{2}} \eta, -(-\Delta)^{\frac{s}{2}} Z_j] \right) \\ & \quad - h \cdot Z_j(1 - \eta) + c(s) Z_0 Z_j(1 - \eta) \\ &= O(R_1^{-\varepsilon}) \end{aligned}$$

for some small positive number ε uniformly on $\tau \in (\tau_0, \tau_1)$. Then (5.13) hold by letting $R_1 \rightarrow +\infty$. Finally, we claim that for all $\tau_1 > 0$ large enough, any ϕ with $\|\phi\|_{a, \tau_1} < +\infty$ solving (5.11) and satisfying (5.13), we have

$$\|\phi\|_{a, \tau_1} \lesssim \|h\|_{2s+a, \tau_1}. \quad (5.14)$$

Hence (5.10) holds.

To prove (5.14), we use the contradiction argument. Suppose that there exist sequences $\tau_1^k \rightarrow +\infty$ and ϕ_k, h_k, c_k satisfying

$$\begin{cases} \partial_\tau \phi_k = -(-\Delta)^s \phi_k + pU^{p-1}(y) \phi_k + h_k - c_k(\tau) Z_0(y), & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \int_{\mathbb{R}^n} \phi_k(y, \tau) \cdot Z_j(y) dy = 0 \quad \text{for all } \tau \in (\tau_0, \tau_1^k), j = 0, 1, \dots, n+1, \\ \phi_k(y, \tau_0) = 0, y \in \mathbb{R}^n \end{cases}$$

and

$$\|\phi_k\|_{a, \tau_1^k} = 1, \quad \|h_k\|_{2s+a, \tau_1^k} \rightarrow 0. \quad (5.15)$$

By (5.12), we have $\sup_{\tau \in (\tau_0, \tau_1^k)} \tau^\nu c_k(\tau) \rightarrow 0$. First, we claim that

$$\sup_{\tau_0 < \tau < \tau_1^k} \tau^\nu |\phi_k(y, \tau)| \rightarrow 0 \quad (5.16)$$

holds uniformly on compact subsets of \mathbb{R}^n . Indeed, if for some $|y_k| \leq M$ and $\tau_0 < \tau_2^k < \tau_1^k$,

$$(\tau_2^k)^\nu |\phi_k(y_k, \tau_2^k)| \geq \frac{1}{2},$$

then it is easy to see that $\tau_2^k \rightarrow +\infty$. Now, we define

$$\tilde{\phi}_n(y, \tau) = (\tau_2^k)^\nu \phi_n(y, \tau_2^k + \tau).$$

Then

$$\partial_\tau \tilde{\phi}_k = L_0[\tilde{\phi}_k] + \tilde{h}_k - \tilde{c}_k(\tau) Z_0(y) \text{ in } \mathbb{R}^n \times (\tau_0 - \tau_2^k, 0],$$

with $\tilde{h}_k \rightarrow 0$, $\tilde{c}_k \rightarrow 0$ uniformly on compact subsets of $\mathbb{R}^n \times (-\infty, 0]$ and

$$|\tilde{\phi}_k(y, \tau)| \leq \frac{1}{1 + |y|^a} \text{ in } \mathbb{R}^n \times (\tau_0 - \tau_2^k, 0].$$

Using the fact that $a \in (2s, n - 2s)$ and the dominant convergence theorem, we have $\tilde{\phi}_k \rightarrow \tilde{\phi}$ uniformly on compact subsets of $\mathbb{R}^n \times (-\infty, 0]$ with $\tilde{\phi} \neq 0$ and

$$\begin{cases} \partial_\tau \tilde{\phi} = -(-\Delta)^s \tilde{\phi} + pU^{p-1}(y)\tilde{\phi} & \text{in } \mathbb{R}^n \times (-\infty, 0], \\ \int_{\mathbb{R}^n} \tilde{\phi}(y, \tau) \cdot Z_j(y) dy = 0 & \text{for all } \tau \in (-\infty, 0], j = 0, 1, \dots, n+1, \\ |\tilde{\phi}(y, \tau)| \leq \frac{1}{1 + |y|^a} & \text{in } \mathbb{R}^n \times (-\infty, 0], \\ \tilde{\phi}(y, \tau_0) = 0, y \in \mathbb{R}^n. \end{cases} \quad (5.17)$$

We claim that $\tilde{\phi} = 0$, which is a contradiction. By fractional parabolic regularity (see [34]), $\tilde{\phi}(y, \tau)$ is smooth. A scaling argument shows

$$(1 + |y|^s) |(-\Delta)^{\frac{s}{2}} \tilde{\phi}| + |\tilde{\phi}_\tau| + |(-\Delta)^s \tilde{\phi}| \lesssim (1 + |y|)^{-2s-a}.$$

Differentiating (5.17), we get $\partial_\tau \tilde{\phi}_\tau = -(-\Delta)^s \tilde{\phi}_\tau + pU^{p-1}(y)\tilde{\phi}_\tau$ and

$$(1 + |y|^s) |(-\Delta)^{\frac{s}{2}} \tilde{\phi}_\tau| + |\tilde{\phi}_{\tau\tau}| + |(-\Delta)^s \tilde{\phi}_\tau| \lesssim (1 + |y|)^{-4s-a}.$$

Moreover, it holds that

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,$$

where

$$B(\tilde{\phi}, \tilde{\phi}) = \int_{\mathbb{R}^n} \left[|(-\Delta)^{\frac{s}{2}} \tilde{\phi}|^2 - pU^{p-1}(y) |\tilde{\phi}|^2 \right] dy.$$

Since $\int_{\mathbb{R}^n} \tilde{\phi}(y, \tau) \cdot Z_j(y) dy = 0$ for all $\tau \in (-\infty, 0]$, $j = 0, 1, \dots, n+1$, $B(\tilde{\phi}, \tilde{\phi}) \geq 0$. Also, we have

$$\int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 = -\frac{1}{2} \partial_\tau B(\tilde{\phi}, \tilde{\phi}).$$

From these relations,

$$\partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 < +\infty.$$

Hence $\tilde{\phi}_\tau = 0$. So $\tilde{\phi}$ is independent of τ and $L_0[\tilde{\phi}] = 0$. Since $\tilde{\phi}$ is bounded, by the nondegeneracy of L_0 (see, [19]), $\tilde{\phi}$ is a linear combination of Z_j , $j = 1, \dots, n+1$. But $\int_{\mathbb{R}^n} \tilde{\phi} \cdot Z_j = 0$, $j = 1, \dots, n$, $\tilde{\phi} = 0$, a contradiction. Thus (5.16) holds.

From (5.15), there exists a certain y_k with $|y_k| \rightarrow +\infty$ such that

$$(\tau_2^k)^\nu (1 + |y_k|^a) |\phi_k(y_k, \tau_2^k)| \geq \frac{1}{2}.$$

Let

$$\tilde{\phi}_k(z, \tau) := (\tau_2^k)^\nu |y_k|^a \phi_k(y_k + |y_k|z, |y_k|^{2s}\tau + \tau_2^k),$$

then

$$\partial_\tau \tilde{\phi}_k = -(-\Delta)^s \tilde{\phi}_k + a_k \tilde{\phi}_k + \tilde{h}_k(z, \tau),$$

where

$$\tilde{h}_k(z, \tau) = (\tau_2^k)^\nu |y_k|^{2s+a} h_k(y_k + |y_k|z, |y_k|^{2s}\tau + \tau_2^k).$$

By the assumption on h_k , one has

$$|\tilde{h}_k(z, \tau)| \lesssim o(1) |\hat{y}_k + z|^{-2s-a} ((\tau_2^k)^{-1} |y_k|^{2s}\tau + 1)^{-\nu}$$

with

$$\hat{y}_k = \frac{y_k}{|y_k|} \rightarrow -\hat{e}$$

and $|\hat{e}| = 1$. Thus $\tilde{h}_k(z, \tau) \rightarrow 0$ uniformly on compact subsets of $\mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]$ and a_k has the same property. Moreover, $|\tilde{\phi}_k(0, \tau_0)| \geq \frac{1}{2}$ and

$$|\tilde{\phi}_k(z, \tau)| \lesssim |\hat{y}_k + z|^{-a} ((\tau_2^k)^{-1} |y_k|^{2s}\tau + 1)^{-\nu}.$$

Hence we may assume $\tilde{\phi}_k \rightarrow \tilde{\phi} \neq 0$ uniformly on compact subsets of $\mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]$ with $\tilde{\phi}$ satisfying

$$\tilde{\phi}_\tau = -(-\Delta)^s \tilde{\phi} \quad \text{in } \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0] \quad (5.18)$$

and

$$|\tilde{\phi}(z, \tau)| \leq |z - \hat{e}|^{-a} \quad \text{in } \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]. \quad (5.19)$$

Similar to Lemma 5.2 of [43], functions $\tilde{\phi}$ satisfying (5.18) and (5.19) must be equal to zero (A proof can be found in [13]), which is a contradiction and we conclude the validity of (5.14). The proof is complete. \square

Proof of Proposition 5.1. First, we consider the problem

$$\begin{cases} \partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau) - c(\tau)Z_0, & y \in \mathbb{R}^n, \tau \geq \tau_0, \\ \phi(y, \tau_0) = 0, & y \in \mathbb{R}^n. \end{cases}$$

Let $(\phi(y, \tau), c(\tau))$ be the unique solution of the nonlocal initial value problem (5.9). From Lemma 5.1, for any $\tau_1 > \tau_0$, we have

$$|\phi(y, \tau)| \lesssim \tau^{-\nu} (1 + |y|)^{-a} \|h\|_{2s+a, \tau_1} \quad \text{for all } \tau \in (\tau_0, \tau_1), y \in \mathbb{R}^n$$

and

$$|c(\tau)| \leq \tau^{-\nu} R^a \|h\|_{2s+a, \tau_1} \quad \text{for all } \tau \in (\tau_0, \tau_1).$$

By assumption, $\|h\|_{2s+a, \nu, \eta} < +\infty$ and $\|h\|_{2s+a, \tau_1} \leq \|h\|_{2s+a, \nu, \eta}$ for an arbitrary τ_1 . It follows that

$$|\phi(y, \tau)| \lesssim \tau^{-\nu} (1 + |y|)^{-a} \|h\|_{2s+a, \nu, \eta} \quad \text{for all } \tau \in (\tau_0, \tau_1), y \in \mathbb{R}^n$$

and

$$|c(\tau)| \leq \tau^{-\nu} R^a \|h\|_{2s+a, \nu, \eta} \quad \text{for all } \tau \in (\tau_0, \tau_1).$$

By the arbitrariness of τ_1 ,

$$|\phi(y, \tau)| \lesssim \tau^{-\nu} (1 + |y|)^{-a} \|h\|_{2s+a, \nu, \eta} \text{ for all } \tau \in (\tau_0, +\infty), y \in \mathbb{R}^n$$

and

$$|c(\tau)| \leq \tau^{-\nu} R^a \|h\|_{2s+a, \nu, \eta} \text{ for all } \tau \in (\tau_0, +\infty).$$

From the regularity result of [42] and a scaling argument, we get the validity of (5.7) and (5.8). \square

5.2. The solvability conditions: choice of the parameters λ and ξ . Denote

$$\lambda(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \vdots \\ \lambda_k(t) \end{pmatrix}, \dot{\lambda}(t) = \begin{pmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \\ \vdots \\ \dot{\lambda}_k(t) \end{pmatrix}, \xi(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_k(t) \end{pmatrix}, \dot{\xi}(t) = \begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \vdots \\ \dot{\xi}_k(t) \end{pmatrix}, q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{pmatrix}.$$

First we consider (5.5) in the case $l = n + 1$.

Lemma 5.2. *When $l = n + 1$, (5.5) is equivalent to*

$$\dot{\lambda}_j + \frac{1}{t} \left(P^T \text{diag} \left(\frac{(2s-1)\bar{\sigma}_r b_r^{2-2s} + 1}{n-4s} \right) P \lambda \right)_j = \Pi_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t) \quad (5.20)$$

where the matrix P , the numbers $\bar{\sigma}_r > 0$ and $b_r > 0$ are defined in Section 2. The right hand side term can be expressed as

$$\begin{aligned} \Pi_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t) &= \frac{t_0^{-\varepsilon}}{R^{a-2s}} \mu_0^{n+1-4s+\sigma} (t) f(t) \\ &\quad + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \Theta \left[\dot{\lambda}, \dot{\xi}, \mu_0^{n-4s}(t) \lambda, \mu_0^{n-4s}(\xi - q), \mu_0^{n+1-4s+\sigma} \phi \right] (t) \end{aligned} \quad (5.21)$$

where $f(t)$ and $\Theta \left[\dot{\lambda}, \dot{\xi}, \mu_0^{n-4s}(t) \lambda, \mu_0^{n-4s}(\xi - q), \mu_0^{n+1-4s+\sigma} \phi \right] (t)$ are smooth and bounded functions for $t \in [t_0, \infty)$. Further, the following estimates hold,

$$\begin{aligned} |\Theta[\dot{\lambda}_1](t) - \Theta[\dot{\lambda}_2](t)| &\lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\dot{\lambda}_1(t) - \dot{\lambda}_2(t)| \\ |\Theta[\dot{\xi}_1](t) - \Theta[\dot{\xi}_2](t)| &\lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\dot{\xi}_1(t) - \dot{\xi}_2(t)|, \\ |\Theta[\mu_0^{n-4s} \lambda_1](t) - \Theta[\mu_0^{n-4s} \lambda_2](t)| &\lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\lambda_1(t) - \lambda_2(t)| \\ |\Theta[\mu_0^{n-4s}(\xi_1 - q)](t) - \Theta[\mu_0^{n-4s}(\xi_2 - q)](t)| &\lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\xi_1(t) - \xi_2(t)|, \\ |\Theta[\mu_0^{n+1-4s+\sigma} \phi_1](t) - \Theta[\mu_0^{n+1-4s+\sigma} \phi_2](t)| &\lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\phi_1(t) - \phi_2(t)\|_{n-2s+\sigma, a}. \end{aligned} \quad (5.22)$$

Proof. Suppose ϕ satisfies (4.15). For a fixed $j \in \{1, \dots, k\}$, we compute

$$\int_{B_{2R}} H_j[\phi, \lambda, \xi, \dot{\lambda}, \dot{\xi}](y, t(\tau)) Z_{n+1}(y) dy,$$

where H_j is given by (5.2). Decompose

$$\begin{aligned}
& \mu_{0j}^{\frac{n+2s}{2}} S_{\mu, \xi, j}(\xi_j + \mu_{0j}y, t) \\
&= \left(\frac{\mu_{0j}}{\mu_j} \right)^{\frac{n+2s}{2}} [\mu_{0j} S_1(z, t) + \lambda_j b_j^{2s-1} S_2(z, t) + \mu_j S_3(z, t)]_{z=\xi_j + \mu_j y} \\
&+ \left(\frac{\mu_{0j}}{\mu_j} \right)^{\frac{n+2s}{2}} \mu_{0j} [S_1(\xi_j + \mu_{0j}y, t) - S_1(\xi_j + \mu_j y, t)] \\
&+ \left(\frac{\mu_{0j}}{\mu_j} \right)^{\frac{n+2s}{2}} \lambda_j b_j^{2s-1} [S_2(\xi_j + \mu_{0j}y, t) - S_2(\xi_j + \mu_j y, t)] \\
&+ \left(\frac{\mu_{0j}}{\mu_j} \right)^{\frac{n+2s}{2}} \mu_j [S_3(\xi_j + \mu_{0j}y, t) - S_3(\xi_j + \mu_j y, t)],
\end{aligned}$$

where

$$\begin{aligned}
& S_1(z) \\
&= (b_j \mu_0)^{2s-2} \dot{\lambda}_j \\
&\times \left(Z_{n+1} \left(\frac{z - \xi_j}{\mu_j} \right) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{\left(1 + \left| \frac{z - \xi_j}{\mu_j} \right|^2 \right)^{\frac{n-2s}{2}}} - 2sApU \left(\frac{z - \xi_j}{\mu_j} \right)^{p-1} \right) \\
&- \mu_0^{n-2s-2} pU(y_j)^{p-1} \sum_{i=1}^k M_{ij} \lambda_i,
\end{aligned}$$

$$\begin{aligned}
& S_2(z) = (2s-1) \mu_0^{2s-2} \dot{\mu}_0 \left(Z_{n+1} \left(\frac{z - \xi_j}{\mu_j} \right) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{\left(1 + \left| \frac{z - \xi_j}{\mu_j} \right|^2 \right)^{\frac{n-2s}{2}}} \right) \\
&+ pU \left(\frac{z - \xi_j}{\mu_j} \right)^{p-1} \mu_0^{n-2s-1} \\
&\times \left(-b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j^{\frac{n-6s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) + (2s-1)B \right)
\end{aligned}$$

and

$$\begin{aligned}
& S_3(z) = \mu_j^{2s-2} \alpha_{n,s} (n-2s) \frac{\dot{\xi}_j \cdot \frac{z - \xi_j}{\mu_j}}{\left(1 + \left| \frac{z - \xi_j}{\mu_j} \right|^2 \right)^{\frac{n-2s}{2} + 1}} + pU \left(\frac{z - \xi_j}{\mu_j} \right)^{p-1} \\
&\times \left(-\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right) \cdot \left(\frac{z - \xi_j}{\mu_j} \right).
\end{aligned}$$

By direct computations, we have

$$\begin{aligned} \int_{B_{2R}} S_1(\xi_j + \mu_j y) Z_{n+1}(y) dy &= (2sAc_1 + c_2)(1 + O(R^{4s-n})) \dot{\lambda}_j (b_j \mu_0)^{2s-2} \\ &\quad + c_1(1 + O(R^{-2s})) \mu_0^{n-2s-2} \sum_{i=1}^k M_{ij} \lambda_i, \end{aligned}$$

$$\begin{aligned} \int_{B_{2R}} S_2(\xi_j + \mu_j y) Z_{n+1}(y) dy &= -(2s-2) \mu_0^{n-2s-1} \frac{2sAc_1 + c_2}{(n-4s)c_{n,s}^{n-4s}} + O(R^{4s-n} + R^{-2s}) \mu_0^{n-2s-1} \\ &= -(2s-2) \mu_0^{n-2s-1} \frac{2sC_1}{(n-2s)} + O(R^{4s-n} + R^{-2s}) \mu_0^{n-2s-1} \end{aligned}$$

and

$$\int_{B_{2R}} S_3(\xi_j + \mu_j y) Z_{n+1}(y) dy = 0 \text{ (by symmetry).}$$

Since $\frac{\mu_{0j}}{\mu_j} = (1 + \frac{\lambda_j}{\mu_{0j}})^{-1}$, for any $l = 1, 2, 3$, we have

$$\begin{aligned} \int_{B_{2R}} [S_l(\xi_j + \mu_{0j} y, t) - S_l(\xi_j + \mu_j y, t)] Z_{n+1}(y) dy \\ = g(t, \frac{\lambda}{\mu_0}) \mu_0^{2s-2} \dot{\lambda}_j + g(t, \frac{\lambda}{\mu_0}) \mu_0^{2s-2} \dot{\xi} + g(t, \frac{\lambda}{\mu_0}) \sum_i \mu_0^{n-2s-2} \lambda_i + \mu_0^{n-2s-1+\sigma} f(t), \end{aligned}$$

where f, g are smooth and bounded functions such that $g(\cdot, s) \sim s$ as $s \rightarrow 0$. Thus

$$\begin{aligned} c \left(\frac{\mu_j}{\mu_{0j}} \right)^{\frac{n+2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} \mu_{0j}^{\frac{n+2s}{2}} S_{\mu, \xi, j}(\xi_j + \mu_{0j} y, t) Z_{n+1}(y) dy \\ = \left[\dot{\lambda}_j + \frac{1}{t} \left(P^T \text{diag} \left(\frac{\frac{n-2s}{2s} \bar{\sigma}_r b_r^{2-2s} + 1}{n-4s} \right) P \lambda \right)_j \right] \\ + \frac{t_0^{-\varepsilon}}{R^{a-2s}} g(t, \frac{\lambda}{\mu_0}) (\dot{\lambda} + \dot{\xi}) + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \mu_0^{n-4s} g(t, \frac{\lambda}{\mu_0}), \end{aligned}$$

where c is a positive number, the function g is smooth, bounded and $g(\cdot, s) \sim s$ as $s \rightarrow 0$.

Next we compute $p \mu_{0j}^{\frac{n-2s}{2}} (1 + \frac{\lambda_j}{\mu_{0j}})^{-2s} \int_{B_{2R}} U^{p-1}(\frac{\mu_{0j}}{\mu_j} y) \psi(\xi_j + \mu_{0j} y, t) Z_{n+1}(y) dy$. The principal part is $I := \int_{B_{2R}} U^{p-1}(y) \psi(\xi_j + \mu_{0j} y, t) Z_{n+1}(y) dy$. Recall $\psi = \psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t)$, we have

$$\begin{aligned} I &= \psi[0, q, 0, 0, 0](q_j, t) \int_{B_{2R}} U^{p-1}(y) Z_{n+1}(y) dy \\ &\quad + \int_{B_{2R}} U^{p-1}(y) Z_{n+1}(y) (\psi[0, q, 0, 0, 0](\xi_j + \mu_{0j} y, t) - \psi[0, q, 0, 0, 0](q_j, t)) dy \\ &\quad + \int_{B_{2R}} U^{p-1}(y) Z_{n+1}(y) (\psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi] - \psi[0, q, 0, 0, 0])(\xi_j + \mu_{0j} y, t) dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By (4.16), $I_1 = \frac{t_0^{-\varepsilon}}{R^{a-2s}} \mu_0^{\frac{n-2s}{2} + \sigma} f(t)$ with f smooth and bounded. By (4.17), $I_2 = \frac{t_0^{-\varepsilon}}{R^{a-2s}} \mu_0^{\frac{n-2s}{2} + \sigma} g(t, \frac{\lambda}{\mu_0}, \xi - q)$ for a smooth and bounded function g satisfying $g(\cdot, s, \cdot) \sim s$ and $g(\cdot, \cdot, s) \sim s$ as $s \rightarrow 0$. From the mean value theorem again, we have

$$\begin{aligned} I_3 = \int_{B_{2R}} U^{p-1}(y) Z_{n+1}(y) & \left[\partial_\lambda \psi[0, q, 0, 0, 0][s\lambda](\xi_j + \mu_{0j}y, t) \right. \\ & + \partial_\xi \psi[0, q, 0, 0, 0][s(\xi_j - q_j)](\xi_j + \mu_{0j}y, t) + \partial_{\dot{\lambda}} \psi[0, q, 0, 0, 0][s\dot{\lambda}](\xi_j + \mu_{0j}y, t) \\ & \left. + \partial_{\dot{\xi}} \psi[0, q, 0, 0, 0][s\dot{\xi}](\xi_j + \mu_{0j}y, t) + \partial_\phi \psi[0, q, 0, 0, 0][s\phi](\xi_j + \mu_{0j}y, t) \right] dy \end{aligned}$$

for some $s \in (0, 1)$. Using Proposition 4.2, I_3 is the sum of terms like

$$\mu_0^{-\frac{n-6s}{2} - 1 + \sigma} \frac{t_0^{-\varepsilon}}{R^{a-2s}} f(t) (\dot{\lambda} + \dot{\xi}) F[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t)$$

and

$$\mu_0^{\frac{n-2s}{2} - 1} \frac{t_0^{-\varepsilon}}{R^{a-2s}} f(t) (\lambda + \xi) F[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t),$$

where f is a smooth, bounded function and F is a nonlocal operator satisfying $F[0, q, 0, 0, 0](t)$ bounded.

Now, we consider the terms $B_j[\phi_j]$, $B_j^0[\phi_j]$ and obtain that

$$\int_{B_{2R}} B_j[\phi_j](y, t) Z_{n+1}(y) dy = \frac{t_0^{-\varepsilon}}{R^{a-2s}} [\mu_0^{n+1-4s+\sigma}(t) \ell[\phi](t) + \dot{\xi}_j \ell[\phi](t)]$$

and

$$\int_{B_{2R}} B_j^0[\phi_j](y, t) Z_{n+1}(y) dy = \frac{t_0^{-\varepsilon}}{R^{a-2s}} \mu_0^{n-2s-1} g\left(\frac{\lambda}{\mu_0}\right) [\phi](t)$$

for a smooth function $g(s)$ satisfying $g(s) \sim s$ as $s \rightarrow 0$, $\ell[\phi](t)$ is smooth and bounded in t . Combining the above estimates, we conclude the result. \square

Similarly, we compute

$$\int_{B_{2R}} H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)) Z_l(y) dy, \quad (5.23)$$

for any $j = 1, \dots, k$, $l = 1, \dots, n$. We have

Lemma 5.3. For $j = 1, \dots, k$, $l = 1, \dots, n$, (5.5) is equivalent to

$$\dot{\xi}_j = \Pi_{2,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \quad (5.24)$$

$$\Pi_{2,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t)$$

$$\begin{aligned} & = \mu_0^{n-4s+2} c \left[b_j^{n-2s} \nabla H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] + \mu_0^{n-4s+2+\sigma}(t) f_j(t) \\ & + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \Theta[\dot{\lambda}, \dot{\xi}, \mu_0^{n-2s-2}(t) \lambda, \mu_0^{n-2s-1}(\xi - q), \mu_0^{n+1-4s+\sigma} \phi](t), \end{aligned}$$

where $c = \frac{p \int_{\mathbb{R}^n} U^{p-1} \frac{\partial U}{\partial y_1} y_1 dy}{\int_{\mathbb{R}^n} \left(\frac{\partial U}{\partial y_1} \right)^2 dy}$, $f_j(t)$ is an n dimensional vector function which is smooth and bounded for $t \in [t_0, \infty)$. The function Θ has the same properties as in Lemma 5.2.

The proof of Lemma 5.3 is similar to that of Lemma 5.2 so we omit it. From Lemma 5.2 and Lemma 5.3, we know that the orthogonality conditions

$$\int_{B_{2R}} H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)) Z_l(y) dy, \text{ for } j = 1, \dots, k \text{ and } l = 1, \dots, n+1,$$

are equivalent to the system of ODEs for λ and ξ

$$\begin{cases} \dot{\lambda}_j + \frac{1}{t} \left(P^T \text{diag} \left(\frac{\frac{n-2s}{2s} \bar{\sigma}_r b_r^{2-2s} + 1}{n-4s} \right) P \lambda \right)_j = \Pi_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \\ \dot{\xi}_j = \Pi_{2,j}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), j = 1, \dots, k. \end{cases} \quad (5.25)$$

System (5.25) is solvable for parameters λ and ξ satisfying (4.12) and (4.13). Indeed, we have

Proposition 5.2. *There exists a solution $\lambda = \lambda[\phi](t)$, $\xi = \xi[\phi](t)$ to (5.25) satisfying (4.12) and (4.13). For $t \in (t_0, \infty)$, it holds that*

$$\mu_0^{-(1+\sigma)}(t) |\lambda[\phi_1](t) - \lambda[\phi_2](t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\phi_1 - \phi_2\|_{n-2s+\sigma, a} \quad (5.26)$$

and

$$\mu_0^{-(1+\sigma)}(t) |\xi[\phi_1](t) - \xi[\phi_2](t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\phi_1 - \phi_2\|_{n-2s+\sigma, a}. \quad (5.27)$$

Proof. Let h be a vector function with $\|h\|_{n+1-4s+\sigma} \lesssim \frac{1}{R^{a-2s}}$. The solution to

$$\dot{\lambda}_j + \frac{1}{t} \left(P^T \text{diag} \left(\frac{\frac{n-2s}{2s} \bar{\sigma}_r b_r^{2-2s} + 1}{n-4s} \right) P \lambda \right)_j = h(t)_j \quad (5.28)$$

can be expressed as

$$\lambda(t) = P^T \nu(t), \nu(t) = \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \vdots \\ \nu_k(t) \end{pmatrix},$$

$$\nu_j(t) = t^{-\frac{1+\frac{n-2s}{2s} \bar{\sigma}_j b_j^{2-2s}}{n-4s}} \left[d_j + \int_{t_0}^t \tau^{\frac{1+\frac{n-2s}{2s} \bar{\sigma}_j b_j^{2-2s}}{n-4s}} (Ph)_j(\tau) d\tau \right], \quad (5.29)$$

where d_j , $j = 1, \dots, k$ are arbitrary constants. Then, for $0 \leq d := \max_{i=1, \dots, k} |d_i|$, we have

$$\|t^{\frac{1+\sigma}{n-4s}} \lambda(t)\|_{L^\infty(t_0, \infty)} \lesssim t_0^{-\frac{\sigma-\sigma}{n-4s}} d + \|h\|_{n+1-4s+\sigma}$$

and

$$\|\dot{\lambda}(t)\|_{n+1-4s+\sigma} \lesssim t_0^{-\frac{\sigma-\sigma}{n-4s}} d + \|h\|_{n+1-4s+\sigma}.$$

Let $\Lambda(t) = \dot{\lambda}(t)$, then

$$\Lambda + \frac{1}{t} \left(P^T \text{diag} \left(\frac{\frac{n-2s}{2s} \bar{\sigma}_r b_r^{2-2s} + 1}{n-4s} \right) \right) P \int_t^\infty \Lambda(s) ds = h(t), \quad (5.30)$$

which defines a linear operator $\mathcal{L}_1 : h \rightarrow \Lambda$ associating to any h with $\|h\|_{n+1-4s+\sigma}$ bounded the solution Λ . \mathcal{L}_1 is continuous between the spaces $L^\infty(t_0, \infty)^k$ with the $\|\cdot\|_{n+1-4s+\sigma}$ -topology.

For any $h : [t_0, \infty) \rightarrow \mathbb{R}^k$ with $\|h\|_{n+1-4s+\sigma}$ bounded, the solution to

$$\dot{\xi}_j = \mu_0^{n-4s+2} c \left[b_j^{n-2s} \nabla H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] + h(t) \quad (5.31)$$

is given by

$$\xi_j(t) = \xi_j^0(t) + \int_t^\infty h(s) ds, \quad (5.32)$$

where

$$\xi_j^0(t) = q_j + c \left[-b_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \int_t^\infty \mu_0^{n-4s+2}(s) ds.$$

Then we have

$$|\xi_j(t) - q_j| \lesssim t^{-\frac{2}{n-4s}} + t^{-\frac{1+\sigma}{n-4s}} \|h\|_{n+1-4s+\sigma}$$

and

$$\|\dot{\xi}_j - \dot{\xi}_j^0\|_{n+1-4s+\sigma} \lesssim \|h\|_{n+1-4s+\sigma}.$$

Let $\Xi(t) = \dot{\xi}(t) - \dot{\xi}^0$ which is a vector function, then (5.32) defines a linear operator $\mathcal{L}_2 : h \rightarrow \Xi$ which is continuous in the $\|\cdot\|_{n+1-4s+\sigma}$ -topology.

Observe that (λ, ξ) is a solution of (5.25) if $(\Lambda = \dot{\lambda}, \Xi = \dot{\xi} - \dot{\xi}^0)$ is a fixed point for the problem

$$(\Lambda, \Xi) = \mathcal{A}(\Lambda, \Xi) \quad (5.33)$$

where

$$\mathcal{A} := \left(\mathcal{L}_1(\hat{\Pi}_1[\Lambda, \Xi, \phi]), \mathcal{L}_2(\hat{\Pi}_2[\Lambda, \Xi, \phi]) \right) = (\bar{A}_1(\Lambda, \Xi), \bar{A}_2(\Lambda, \Xi))$$

with

$$\hat{\Pi}_1[\Lambda, \Xi, \phi] := \Pi_1 \left[\int_t^\infty \Lambda, q + \int_t^\infty \Xi, \Lambda, \Xi, \phi \right],$$

and

$$\hat{\Pi}_2[\Lambda, \Xi, \phi] := \Pi_2 \left[\int_t^\infty \Lambda, q + \int_t^\infty \Xi, \Lambda, \Xi, \phi \right].$$

Let

$$K := R^{a-2s} \max\{\|f\|_{n+1-4s+\sigma}, \|f_1\|_{n+1-4s+\sigma}, \dots, \|f_k\|_{n+1-4s+\sigma}\}$$

where f, f_1, \dots, f_k are defined in Lemma 5.2 and Lemma 5.3. Now, we show that problem (5.33) has a fixed point (Λ, Ξ) in the following space

$$\mathcal{B} = \left\{ (\Lambda, \Xi) \in L^\infty(t_0, \infty) \times L^\infty(t_0, \infty) : \|\Lambda\|_{n-2s-1+(2s-1)\sigma} + \|\Xi\|_{n-2s-1+(2s-1)\sigma} \leq \frac{cK}{R^{a-2s}} \right\}$$

for suitable $c > 0$. Indeed, from (5.21) we have

$$\begin{aligned} & \left| t^{\frac{n+1-4s+\sigma}{n-4s}} \bar{A}_1(\Lambda, \Xi) \right| \\ & \lesssim t_0^{-\frac{\sigma-\sigma}{n-4s}} d + \frac{1}{R^{a-2s}} \|\phi\|_{n-2s+\sigma, a} + \frac{K}{R^{a-2s}} \\ & \quad + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\Lambda\|_{n+1-4s+\sigma} + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\Xi\|_{n+1-4s+\sigma} \end{aligned}$$

and

$$\begin{aligned} \left| t^{\frac{n+1-4s+\sigma}{n-4s}} \bar{A}_2(\Lambda, \Xi) \right| &\lesssim \frac{1}{R^{a-2s}} \|\phi\|_{n-2s+\sigma, a} + \frac{K}{R^{a-2s}} \\ &\quad + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\Lambda\|_{n+1-4s+\sigma} + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\Xi\|_{n+1-4s+\sigma}. \end{aligned}$$

Thus, for d satisfying $t_0^{-\frac{\bar{\sigma}-\sigma}{n-4s}} d < \frac{K}{R^{a-2s}}$ and the constant c chosen sufficiently large, $\mathcal{A}(\mathcal{B}) \subset \mathcal{B}$. As for the Lipschitz property of \mathcal{A} , we have

$$\begin{aligned} &t^{\frac{n+1-4s+\sigma}{n-4s}} \left| \bar{A}_1(\Lambda_1, \Xi) - \bar{A}_1(\Lambda_2, \Xi) \right| \\ &= t^{\frac{n+1-4s+\sigma}{n-4s}} \left| \mathcal{L}_1(\hat{\Pi}_1[\Lambda_1, \Xi, \phi] - \hat{\Pi}_1[\Lambda_2, \Xi, \phi]) \right| \\ &\leq t^{\frac{n+1-4s+\sigma}{n-4s}} t_0^{-\varepsilon} \left| \mathcal{L}_1(\Theta_2(\Lambda_1, \Xi) - \Theta_2(\Lambda_2, \Xi)) \right| \\ &\quad + t^{\frac{n+1-4s+\sigma}{n-4s}} t_0^{-\varepsilon} \left| \mathcal{L}_1(\mu_0^{n-2s-2} \Theta_3(\Lambda_1, \Xi) - \mu_0^{n-2s-2} \Theta_3(\Lambda_2, \Xi)) \right| \\ &\leq t_0^{-\varepsilon} \|\Lambda_1 - \Lambda_2\|_{n+1-4s+\sigma}. \end{aligned}$$

The same estimate holds for $|\bar{A}_1(\Lambda, \Xi_1) - \bar{A}_1(\Lambda, \Xi_2)|$. Thus, we have

$$\|\mathcal{A}(\Lambda_1, \Xi_1) - \mathcal{A}(\Lambda_2, \Xi_2)\|_{n+1-4s+\sigma} \leq t_0^{-\varepsilon} \|\Lambda_1 - \Lambda_2\|_{n+1-4s+\sigma}.$$

Since $t_0^{-\varepsilon} < 1$ when t_0 is large enough, \mathcal{A} is a contraction map. Hence, from the Contraction Mapping Theorem, there exists a solution to system (5.25) with λ, ξ satisfying (4.12) and (4.13).

To prove (5.26) and (5.27), we observe that $\bar{\lambda} = \lambda[\phi_1] - \lambda[\phi_2]$ and $\bar{\xi} = \xi[\phi_1] - \xi[\phi_2]$ satisfy

$$\dot{\lambda} + \frac{1}{t} \left(P^T \text{diag} \left(\frac{\frac{n-2s}{2s} \bar{\sigma}_r b_r^{2-2s} + 1}{n-4s} \right) P \lambda \right) = \bar{\Pi}_1(t), \quad \dot{\xi}_j = \bar{\Pi}_{2,j}(t), \quad j = 1, \dots, k$$

where

$$\begin{aligned} &(\bar{\Pi}_1(t))_j \\ &= c p \mu_j^{\frac{n-2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} U^{p-1} \left(\frac{\mu_{0j}}{\mu_j} y \right) [\psi[\phi_1] - \psi[\phi_2]] (\xi_j + \mu_{0j} y, t) Z_{n+1}(y) dy \\ &\quad + c \left(\frac{\mu_j}{\mu_{0j}} \right)^{\frac{n+2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} [B_j[(\phi_1)_j] - B_j[(\phi_2)_j]] Z_{n+1}(y) dy \\ &\quad + c \left(\frac{\mu_j}{\mu_{0j}} \right)^{\frac{n+2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} [B_j^0[(\phi_1)_j] - B_j^0[(\phi_2)_j]] Z_{n+1}(y) dy \end{aligned}$$

and

$$\begin{aligned} &(\bar{\Pi}_1(t))_j \\ &= c p \mu_j^{\frac{n-2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} U^{p-1} \left(\frac{\mu_{0j}}{\mu_j} y \right) [\psi[\phi_1] - \psi[\phi_2]] (\xi_j + \mu_{0j} y, t) \frac{\partial U}{\partial y_j}(y) dy \\ &\quad + c \left(\frac{\mu_j}{\mu_{0j}} \right)^{\frac{n+2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} [B_j[(\phi_1)_j] - B_j[(\phi_2)_j]] \frac{\partial U}{\partial y_j}(y) dy \\ &\quad + c \left(\frac{\mu_j}{\mu_{0j}} \right)^{\frac{n+2s}{2}} \mu_{0j}^{1-2s} \int_{B_{2R}} [B_j^0[(\phi_1)_j] - B_j^0[(\phi_2)_j]] \frac{\partial U}{\partial y_j}(y) dy. \end{aligned}$$

Then (5.26) and (5.27) follow from (5.22). This completes the proof. \square

6. GLUING: PROOF OF THEOREM 1.1

After we have chosen parameters $\lambda = \lambda[\phi]$ and $\xi = \xi[\phi]$ such that the orthogonality conditions (5.5) hold, we only need to solve problem (5.3) in the class of functions with $\|\phi\|_{a,\nu}$ (or equivalently $\|\phi\|_{n-2s+\sigma,a}$) bounded. With the chosen parameters, we can apply Proposition 5.1 which states that there exists a linear operator \mathcal{T} associating any function $h(y, \tau)$ with $\|h\|_{2s+a,\nu}$ -bounded the solution to (5.6). Thus problem (5.3) is reduced to a fixed point problem

$$\phi = (\phi_1, \dots, \phi_k) = \mathcal{A}(\phi) := (\mathcal{T}(H_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]), \dots, \mathcal{T}(H_k[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi])). \quad (6.1)$$

We claim that, for each $j = 1, \dots, k$, there hold

$$(1 + |y|^\eta) \left[H[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](\cdot, t) \right]_{\eta, B_1(0)} \chi_{B_{2R}(0)}(y) + \left| H[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t) \right| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2s+\sigma}}{1 + |y|^{2s+a}} \quad (6.2)$$

and

$$(1 + |y|^\eta) \left[H[\phi^{(1)}](\cdot, t) - H[\phi^{(2)}](\cdot, t) \right]_{\eta, B_1(0)} \chi_{B_{2R}(0)}(y) + \left| H[\phi^{(1)}] - H[\phi^{(2)}] \right|(y, t) \lesssim t_0^{-\varepsilon} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma,a}. \quad (6.3)$$

From (6.2) and (6.3), \mathcal{A} has a fixed point ϕ within the set of functions $\|\phi\|_{n-2s+\sigma,a} \leq ct_0^{-\varepsilon}$ for some large positive constant c . This proves the existence part of Theorem 1.1.

Estimate (6.2) is obtained from the definition of H_j , Lemma 2.2 and (4.16). As for (6.3), from (5.26) and (5.27), we have

$$\mu_{0j}^{\frac{n+2s}{2}} \left| S_{\mu_1, \xi_{1,j}}(\xi_{j,1} + \mu_{0j}y, t) - S_{\mu_2, \xi_{2,j}}(\xi_{j,2} + \mu_{0j}y, t) \right| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2s+\sigma}(t)}{1 + |y|^{2s+a}} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma,a}$$

where

$$\mu_i = \mu[\phi^{(i)}], \quad \xi_i = \xi[\phi^{(i)}], \quad \xi_{j,i} = \xi_j[\phi^{(i)}], \quad i = 1, 2.$$

By Proposition 4.2, it holds that

$$p\mu_{0j}^{\frac{n-2s}{2}} \left| \frac{\mu_{0j}^{2s}}{\mu_{j,1}^{2s}} U^{p-1} \left(\frac{\mu_{0j}}{\mu_{j,1}} y \right) \psi[\phi^{(1)}](\xi_{j,1} + \mu_{0j}y, t) - \frac{\mu_{0j}^{2s}}{\mu_{j,2}^{2s}} U^{p-1} \left(\frac{\mu_{0j}}{\mu_{j,2}} y \right) \psi[\phi^{(2)}](\xi_{j,2} + \mu_{0j}y, t) \right| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2s+\sigma}(t)}{1 + |y|^{2s+a}} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma,a}$$

where

$$\mu_{j,i} = \mu_j[\phi^{(i)}], \quad \psi[\phi^{(i)}] = \Psi[\lambda_i, \xi_i, \dot{\lambda}_i, \dot{\xi}_i, \phi^{(i)}], \quad i = 1, 2.$$

Finally, from the definitions (3.9) and (3.10) in Section 3,

$$\left| B_j[\phi_j^{(1)}] - B_j[\phi_j^{(2)}] \right| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2s+\sigma}(t)}{1 + |y|^{2s+a}} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma,a}$$

and

$$\left| B_j^0[\phi_j^{(1)}] - B_j^0[\phi_j^{(2)}] \right| \lesssim t_0^{-\varepsilon} \frac{\mu_0^{n-2s+\sigma}(t)}{1 + |y|^{2s+a}} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma, a}$$

hold. This proves the estimate (6.3).

The stability part of Theorem 1.1 is the same as [17], so we omit it. \square

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